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Short Communication

Constant Affine Velocity Predicts the 1/3 Power Law of Planar Motion Perception and Generation

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Numerous studies have shown that the power of 1/3 is important in relating Euclidean velocity to radius of curvature (*R*) in the generation and perception of planar movement. Although the relation between velocity and curvature is clear and very intuitive, no valid explanation for the specific 1/3 value has yet been found. We show that if instead of computing the Euclidean velocity we compute the affine one, a velocity which is invariant to affine transformations, then we obtain that the unique function of *R* which will give (constant) affine invariant velocity is precisely $R^{1/3}$. This means that the 1/3 power law, experimentally found in the studies of hand-drawing and planar motion perception, implies motion at constant affine velocity. Since drawing/perceiving at constant affine velocity implies that curves of equal affine length will be drawn in equal time, we performed an experiment to further support this result. Results showed agreement between the 1/3 power law and drawing at constant affine velocity. Possible reasons for the appearance of affine transformations in the generation and perception of planar movement are discussed. Copyright © 1996 Elsevier Science Ltd

INTRODUCTION

When humans draw planar curves, the instantaneous tangential velocity of the hand decreases as the curvature increases (Abend *et al.*, 1982; Morasso, 1981; Viviani & Terzuolo, 1982). This relationship is best described as a power law where velocity is proportional to the 1/3 power of the radius of curvature (Lacquaniti *et al.*, 1983). This power law has been found in a variety of drawing tasks (Massey *et al.*, 1992; Viviani & Flash, 1995; Viviani & Cenzato, 1985; Viviani & McCollum, 1983; Wann *et al.*, 1988) and has been shown to evolve in the development of drawing skills (Viviani & Schneider, 1991). In addition to describing the production of planar drawing movements, it has been found that the identical power law is involved in the perception of smooth planar motion (Viviani & Stucchi, 1989, 1992). A point-light moving on a plane is perceived as moving with constant velocity when its real velocity holds this same 1/3 power law. Although it has been shown that principles of motion planning such as minimum jerk (Flash & Hogan, 1985) can reproduce similar power law relations in the production of movement (Viviani & Flash, 1995; Wann *et al.*, 1988), these results are of limited use in explaining why a similar power law relation holds in the perception of movement. Moreover, it has been found that the 1/3 power law is not obtained for the gently curved paths generated by the motor system in executing movements which are constrained by only their start and end points (Pollick & Ishimura, 1996). In this paper we offer a fundamental interpretation which describes the inherent duality of the 1/3 power law.

When considering how to account for this 1/3 power we note that although the physical world which we see and manipulate can be described by Euclidean geometry, there is reason to doubt that properties such as Euclidean distance and angles are faithfully reproduced in our internal representations. For example, judgments of static form show that the structure of human visual space (Indow, 1991; Luneberg, 1900) as well as motor space (Fasse *et al.*, 1995) deviate from Euclidean geometry. In addition to these deviations stands the fact that regularities and invariances of the relations between internal representations and the physical world need not be expressed in Euclidean geometry. And in this paper we show that the power law relating figural and kinematic aspects of movement—that Euclidean tangential velocity V_e is proportional to the radius of curvature R to the 1/3 power—can be explained by examination of the affine space rather than the Euclidean one.

Why affine? In vision, affine transformations are obtained when a planar object is rotated and translated in space, and then projected into the eye (camera) via a parallel projection. This is a good model of the human visual system when the object is flat enough, and away from the eye, as in the case of drawing and planar point

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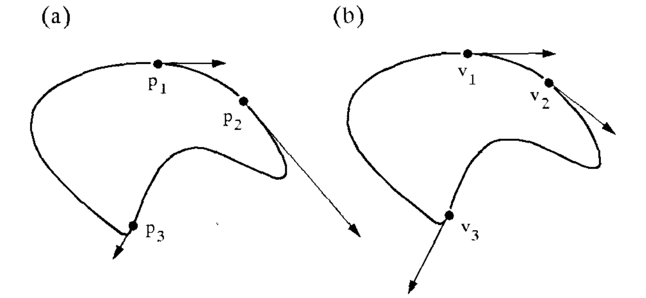

FIGURE 1. Geometry of a planar curve. In (a) a curve parametrized by **p** is given. Note that the tangent vectors have different lengths, since the parametrization **p** is arbitrary. This tangent vectors represent the curve traveling velocity. In (b), the same geometric curve is presented, with a different parametrization. In this case, the parametrization is given by the Euclidean arc-length **v**, which means that the curve is traveled with constant velocity. This makes the tangent vectors equal in length. Although tangent vectors are different in both curves, since the trajectories have different velocities, both curves have the same trace.

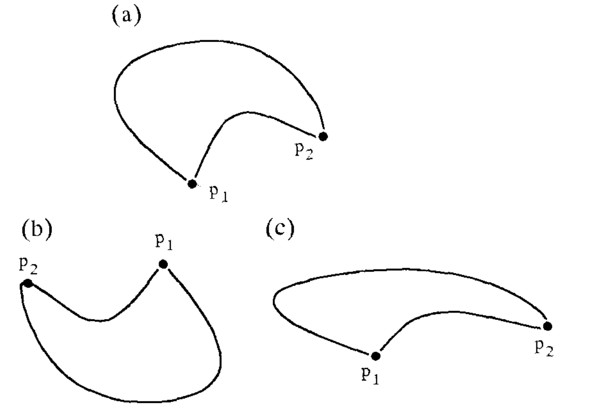

FIGURE 2. Curves related by Euclidean (a) and (b) and affine transformations (a) and (c). While the Euclidean distance between corresponding points in (a) and (b) is preserved, this is not so between (b) and (c). In this case, the affine distance is the preserved one. The 1/3 power predicts that the traveling time from $\mathbf{p}_1$ to $\mathbf{p}_2$ is the same since the figures are related by affine transformations.

movements. Accordingly, affine concepts have been investigated in the analysis of image motion and the perception of three-dimensional structure from motion (Beusmans, 1993; Eagleson, 1992; Koenderink & van Doorn, 1991; Norman & Todd, 1993; Pollick, 1996; Todd & Bressan, 1990) as well as the recognition of planar form (Wagemans *et al.*, 1994). Another way that affine invariance could arise is that the transforms from visual input to motor output could approximate the true Euclidean transformations (Flanders *et al.*, 1992) and do so with affine approximations. Although in this work we do not attempt to isolate the stage in visuo-motor processing at which the affine geometry enters, the essential explanation of the 1/3 power remains the same. Further details on the possible significance of an affine representation and affine constant velocity will be given in the final section, after presenting the theoretical and experimental results.

AFFINE VELOCITY AND CONSTANT PLANAR MOTION

We proceed now to explain the 1/3 power law experimental findings based on differential geometry. A planar curve may be regarded as the trajectory of a point $p \in [0,1]$ on the plane. For each value of p, a point $C(p) = [x(p), y(p)] \in \mathbf{R}^2$ on the curve is obtained (Fig. 1).

The velocity of the trajectory is given by the tangent vector $\frac{\partial C}{\partial p}$. Different parametrizations p give different velocities, but define the same trace or geometric curve. That means that given an increasing function $q(p)$: $\mathbf{R}^+ \to \mathbf{R}^+$, although the traveling velocities are different since $\frac{\partial C}{\partial p} \neq \frac{\partial C}{\partial q}$, the curve $C(q)$ defines the same trace as $C(p)$. Figure 1 presents a picture explaining these concepts.

An important parametrization is the *Euclidean arc-length v* (Spivak, 1979), which means that the curve is traveled with constant velocity, that is $||\frac{\partial C}{\partial v}|| \equiv 1$ (see Fig. 1). Here, $||\cdot||$ represents the classical Euclidean vector norm given by $<\cdot,\cdot>^{1/2}$. In order to transform from an arbitrary parametrization p to this Euclidean arc-length, the operation P defined as

$$v(p) = \int_0^p ||\frac{\partial C(q)}{\partial q}||dq, \tag{1}$$

is used, and the new arc-length parametrized curve is given by $C(P)$ (recall that both curves differ in parametrization but represent the same geometric structure). From the operator P above it is easy to see that since

$$\frac{\partial C}{\partial v} = \frac{\partial C}{\partial p}\frac{\partial p}{\partial v},$$

then $||\frac{\partial C}{\partial v}|| \equiv 1$, obtaining the required constant traveling velocity.

In this case, using the Euclidean parametrization v, the Euclidean length of a curve between v_0 and v_1 is

$$l_e(v_0, v_1) := \int_{v_0}^{v_1} dv.$$

This *Euclidean arc-length* parametrization is invariant with respect to rotations and translations (Euclidean transformations). This means the following: assume $\tilde{C}$ is obtained from C via a rotation and a translation, i.e.,

$$\tilde{C} = \mathbf{R}C + \mathbf{T},$$

where $\mathbf{R}$ is a 2×2 rotation matrix and $\mathbf{T}$ is a 2×1 translation vector. Let v_0 and v_1 be two points in C and $\tilde{v}_0$ and $\tilde{v}_1$ their corresponding points after the transformation **(R,T)** (see Fig. 2.) Then, the Euclidean invariance of the arc-length v gives that $dv = d\tilde{v}$, meaning that distances measured via d are preserved; $l_e(v_0, v_1) = l_e(\tilde{v}_0, \tilde{v}_1)$.

Having the definition of Euclidean arc-length and length, we can define the *Euclidean velocity* via

$$V_e := \frac{dv}{dt},$$

where t stands for time. This is the classical definition of velocity, which relates the (Euclidean) distance l_e

traveled with the time it takes to travel it. Since l_e is invariant to rotations and translations, so is V_e. This velocity V_e is the one measured in the hand-drawing and perception of planar motion experiments. From these empirical studies it was found that subjects draw, under regular conditions, with velocity given by

$$V_e = cR^{1/3}, \tag{2}$$

where c is a constant and R is the radius of curvature. For the case of motion perception, when a point-light moves on a plane with velocity given by Eq. (2), and only with this 1/3 power velocity, subjects report perception of uniform motion. Recall that the radius of curvature $R(p)$ is defined as the radius of the circle that best approximates the curve C at the point p. This radius R is also the inverse of the curvature κ, defined as the rate of change of the unit tangent vector $\vec{T}$, that is

$$\kappa := \|\frac{\partial \vec{T}}{\partial v}\|.$$

Suppose now that instead of only rotations and translations, we allow *affine* transformations, which means that the curve can be stretched with different values in the horizontal and vertical directions. An affine transformation of a curve C is formally defined as

$$\tilde{C} = AC + \mathbf{T},$$

where A is a 2×2 non-singular matrix* and $\mathbf{T}$ is a translation vector as before. For the affine group, the Euclidean arc-length v is not invariant any more, $dv \neq d\tilde{v}$ and $l_e(v_0, v_1) \neq l_e(\tilde{v}_0, \tilde{v}_1)$. We can define a new notion of *affine arc-length* (s), and based on it an *affine length* (l_a), which are affine invariant (Blaschke, 1923; Sapiro & Tannenbaum, 1994). The affine arc-length is given by the requirement

$$\left|\frac{\partial C}{\partial s} \times \frac{\partial^2 C}{\partial s^2}\right| \equiv 1,$$

which means that the area† of the parallelogram determined by the vectors $\frac{\partial C}{\partial s}$ and $\frac{\partial^2 C}{\partial s^2}$ is constant. This gives the simplest affine invariant parametrization.‡

As in the Euclidean case, having a curve C parametrized with an arbitrary parametrization p, to re-parametrize it now in affine arc-length, we use the relation

$$s(p) = \int_0^p \left[\frac{\partial C}{\partial q} \times \frac{\partial^2 C}{\partial q^2}\right]^{1/3} dq. \tag{3}$$

Expressing again the partial derivatives $\frac{\partial C}{\partial s}$ and $\frac{\partial^2 C}{\partial s^2}$ as derivatives on p, and using the relation (3), it is easy to show that the curve re-parametrized to s now holds the affine invariant requirement $|\frac{\partial C}{\partial s} \times \frac{\partial^2 C}{\partial s^2}| \equiv 1$.

*We assume that the determinant of A is equal to 1.

†Length is a non-affine invariant, but area is.

‡Affine differential geometry is not defined at inflection points ($R = \infty$) and thus the definitions are correct only for non-inflection points. However, since inflection points are affine invariant, that is, preserved via an affine transformation, this causes no problems. Simplest here refers to minimal order (number of derivatives).

It is also straightforward to prove that the parametrization s is affine invariant, since $[AV \times AU] = A[V \times U]$, for $U, V \in \mathbf{R}^2$ and A a non-singular 2×2 matrix. Since this parametrization contains the minimal possible number of derivatives for the affine group (Olver, 1995), it is the simplest one (Blaschke, 1923). Any other affine invariant parametrization will be a function of this one or of higher order.

Based on this new parametrization, we define the affine invariant distance between two points s_0, s_1 on the curve C as

$$l_a(s_0, s_1) := \int_{s_0}^{s_1} ds,$$

and the affine velocity as

$$V_a := \frac{ds}{dt}.$$

The affine velocity relates the affine distance l_a with the time it takes to travel it, and both l_a and V_a are affine invariant (see Fig. 2).

As we pointed out before, parametrizations only describe the velocity the curves are traveled, and define the same geometric curve or trace. It is possible in general to transform a curve $C(p)$ parametrized by p into another one parametrized by q, with q being a function of p. This process is called re-parametrization. The formulas for performing this re-parametrization when $q = v$ or $q = s$ are given by Eqs (1) and (3). Assume now that the curve is originally parametrized via Euclidean arc-length v, and we want to re-parametrize it by affine arc-length s. Then, using the relation between an arbitrary parametrization and s given by Eq. (3) (Blaschke, 1923; Sapiro & Tannenbaum, 1994) we have

$$\frac{ds}{dv} = \left|\frac{\partial C}{\partial v} \times \frac{\partial^2 C}{\partial v^2}\right|^{1/3} = |\vec{T} \times \kappa \vec{\mathbf{N}}|^{1/3} = \kappa^{1/3}, \tag{4}$$

where $\vec{T}$, $\vec{N}$, and $\kappa = 1/R$ are the unit tangent, unit normal, and the Euclidean curvature, respectively. In the expression above we used classical relations of differential geometry,

$$\frac{\partial C}{\partial v} = \vec{T}, \quad \frac{\partial^2 C}{\partial v^2} = \kappa \vec{N},$$

together with the fact that $[\vec{T} \times \vec{N}] = 1$.

Based on Eq. (4) we obtain

$$\begin{aligned} V_a &= \frac{ds}{dt} \\ &= \frac{ds}{dv}\frac{dv}{dt} \\ &= \kappa^{1/3} V_e = \frac{1}{R^{1/3}} V_e. \end{aligned}$$

This is the general formula that relates Euclidean velocity to affine velocity. For the case of hand-drawing and planar motion perception [Eq. (2)] we have then

$$V_a \propto c, \tag{5}$$

which means that the curve is traveled with constant affine velocity. This means for example that a circle and an ellipse will be traveled at times proportional to c, since they are related by an affine transformation. Looking at Fig. 2, the 1/3 power law predicts that the drawing times from p_1 and p_2 in Fig. 2(b) and (c) are the same, since both curves are related by an affine transformation.

From Eq. (5) we conclude that traveling with velocity proportional to the 1/3 power of the radius of curvature means that the affine velocity is constant. Moreover, it is easy to prove that the unique function of R that will give (constant) affine velocity is this 1/3 power. This is because it is the unique one that will eliminate the dependence on κ, which is not affine invariant. This means that the 1/3 power is the unique function of curvature which provides that two curves related by an affine transformation are drawn in the same time. The same is true for a point-light moving on two planar trajectories related by affine transformations.

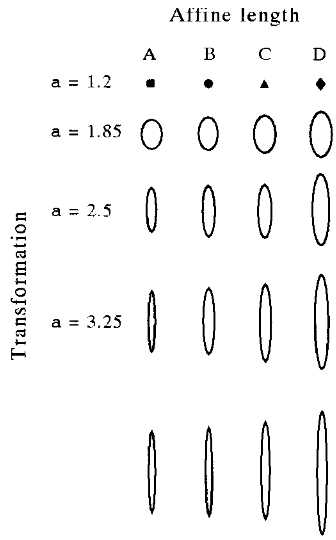

FIGURE 3. The 16 shapes used in the drawing experiment. Each column contains four curves with equal affine length and corresponds to an affine-transformed hippopede. The rows were obtained by stretching an amount α in the vertical direction while compressing by an amount $\frac{1}{\alpha}$ in the horizontal direction.

EXPERIMENT

We performed an experiment to determine if, as predicted, curves were drawn at constant affine velocity and that drawing time remained constant for shapes of equal affine length. Given that the 1/3 power law is both supported by a large body of empirical results and, as described in the previous section, is equivalent to motion at constant affine velocity, we can expect that shapes of equal affine length will be drawn in equal time. However, by recasting the experiment in terms of affine geometry it was possible to illustrate the usefulness of affine concepts in describing drawing movements and their deviations from ideal performance.

Methods

Subjects. Six subjects from the lab subject pool volunteered to participate in the experiment. All subjects were right-handed and naïve to the purpose of the experiment.

Design. Two independent variables were examined: affine length (four different shapes each with a different affine length) and amount of affine transformation. We used an affine transformation which preserved length and was parametrized by a single variable α. Each shape was presented at four values of α. The two variables and the levels of each are illustrated in Fig. 3.

Stimuli. The shapes of the 16 stimuli (Fig. 3) were obtained by the affine transformation of a hippopede (Lawrence, 1972). The polar equation of a hippopede is $r^2 = 4b(a - b\sin^2)$ and the four hippopedes were obtained with values $a = 4.3$ mm and $b = \frac{a}{5}, \frac{a}{4}, \frac{a}{3.25}, \frac{a}{2}$ for the columns left to right (note that in Fig. 3 each shape is rotated so that its long axis is vertically aligned). The affine transformation used to obtain the four rows preserved the affine length of the curve. The form of the transformation was to stretch by an amount α in the vertical direction and compress by an amount $\frac{1}{\alpha}$ in the horizontal direction. The four rows, from top to bottom, correspond to values of $\alpha = 1.2$, 1.85, 2.5 and 3.25. The calculation and generation of curves was performed in Mathematica.

Apparatus. The 16 plane curves were presented on a sheet of A4 paper which was placed on top of a digitizing pad (Wacom SD-312). The spatial accuracy of the pad was 0.02 mm and movement data were collected at a rate of 205 Hz. The digitizing pad was connected to a workstation which was used to store the data and prompt the beginning and end of drawing movements.

Procedure. Subjects participated in a single session in which they twice traced each of the 16 plane curves using a stylus held in their right hand. Each of these 32 drawing epochs lasted 45 sec and was prompted by a start and stop tone from the computer workstation. For each drawing epoch, the movement began at the 12 o'clock position and proceeded in a clockwise direction. The session consisted of two blocks separated by a rest period of 5 min. In each block the entire set of 16 curves was drawn. The order of drawing was arranged so that for each affine length half the curves were drawn by ascending order of α and half by descending order of α.

Subjects were seated at a table and performed the drawing motions in the horizontal plane. Subjects were instructed to accurately draw the curves and told that there was no time constraint. The stylus used to trace the curves left a mark on the sheet of paper and thus subjects had feedback on the accuracy of their motions.

Data processing. For each subject the raw data was first smoothed by a double pass of a fifth-order butterworth filter (low-pass cut-off frequency of 10 Hz) and then differentiated using central-difference equations to find velocity and acceleration. Following this, a fixed number of complete revolutions was extracted from each of the 32 drawing epochs and used for subsequent analysis. The following text explains the extraction process in greater detail. The 32 epochs of each

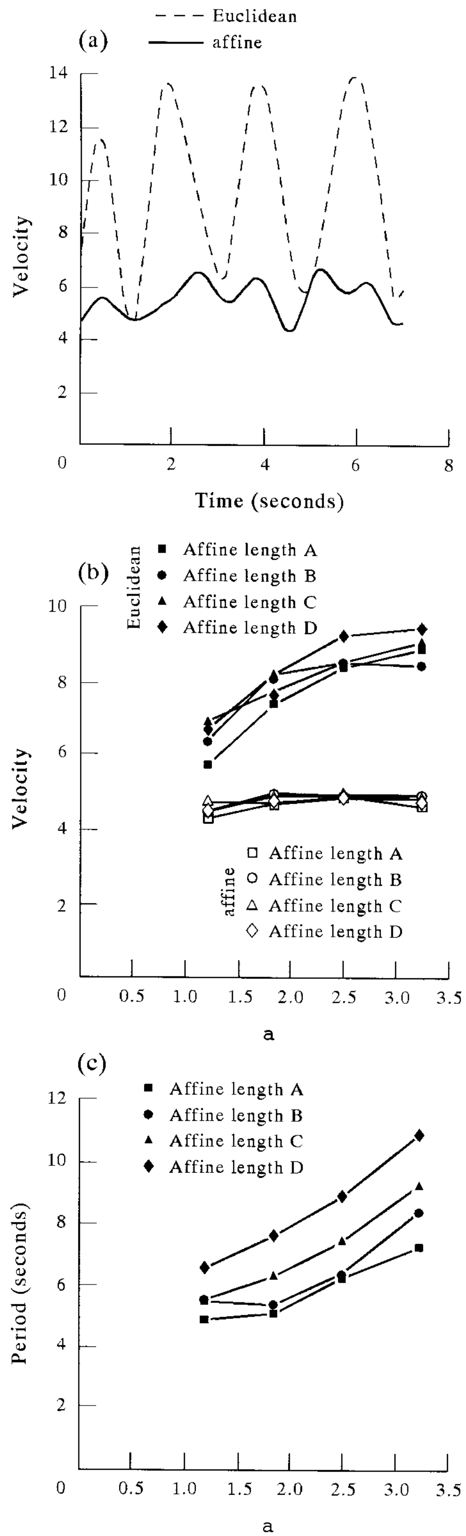

FIGURE 4. (a) An example of corresponding instantaneous Euclidean and affine velocities (filtered at 1 Hz cutoff). Euclidean velocity is periodic with the drawing motion, while affine velocity is roughly constant (units of velocity: Euclidean $\left(\frac{mm}{sec}\right)$, affine $\left(\frac{mm^{2/3}}{sec}\right)$). (b) Averages of subjects' instantaneous affine and Euclidean velocities. Average instantaneous affine velocity is shown in open marks and average instantaneous Euclidean velocity is shown in filled marks (units of velocity: Euclidean $\left(\frac{mm}{sec}\right)$, affine $\left(\frac{mm^{2/3}}{sec}\right)$). (c) Average drawing time did not remain constant for shapes of equal affine length, but increased for shapes with greater Euclidean perimeter. See text and Fig. 5.

individual subject were examined to find the epoch which contained the smallest number of complete revolutions. This number of complete revolutions was then extracted from each of the 32 epochs for further data analysis. The number of complete revolutions obtainable varied between subjects and was, on average, 2.2. Given that each of the 16 shapes was drawn twice this provided, on average, 4.4 complete revolutions as the basis of the subsequent calculations.

RESULTS AND DISCUSSION

The data were first examined to see if a 1/3 power law was obtained. Exponents were obtained for the 16 conditions by taking the extracted data and regressing the logarithm of Euclidean velocity vs the logarithm of the radius of curvature and performing a linear regression (the slope of the regression is the exponent). An analysis of variance (ANOVA) was calculated on the exponents using affine length and affine transformation (α) as factors. Results showed a main effect for the factor of affine transformation $F(3,15) = 6.0$, $P < 0.05$. For the increasing levels of α, the corresponding exponents were 0.24 (0.02), 0.28 (0.01), 0.30 (0.01) and 0.29 (0.01), standard errors of the mean (SEM) indicated in parentheses.

Next, Euclidean drawing velocities were compared to affine drawing velocities, and the total drawing times were examined. The comparison of velocities can be seen in Fig. 4(a) for the instantaneous velocity of a typical drawing movement and in Fig. 4(b) for the instantaneous velocity averaged over complete drawing cycles. It can be seen that compared to Euclidean velocity, affine velocity was roughly constant. However, examination by ANOVA of the effects of affine length and affine transformation (α) on the average instantaneous affine velocity did yield an effect of affine transformation, $F(3,15) = 5.9$, $P < 0.05$. For the different increasing levels of α, the corresponding affine velocities were 4.5 (0.3), 4.8 (0.3), 4.9 (0.3) and 4.8 (0.3); SEM indicated in parentheses. Analysis of drawing times [Fig. 4(c)] revealed that drawing times lengthened with increasing levels of α and appeared correlated to an increase in Euclidean length of the drawn curve. Thus, contrary to prediction, objects with the same affine length were not drawn in equal time. However, the fact that drawing time increased even though affine velocity was constant leads to the prediction that subjects did not accurately draw the presented shape, but instead made errors which resulted in a drawn curve of increased affine length. For this reason we explored the data for errors in reproducing the presented shape.

As a preliminary check, we first explored whether subjects made systematic errors in reproducing the Euclidean length of the presented shape. To accomplish this, we examined the total length (perimeter) of the drawing motions. Results showed that subjects reproduced the Euclidean perimeter with an average error of 0.6 mm (standard deviation 1.3 mm) and that this error

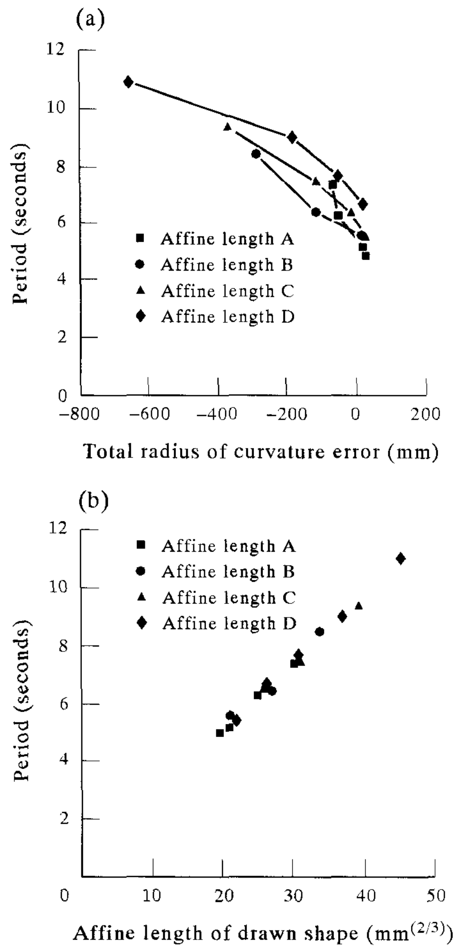

FIGURE 5. (a) A plot of the drawing times vs the average error in the total radius of curvature. This error was defined as the sum of the radius of curvature of the drawn shape minus the approximate numerical integral of the radius of curvature of the presented shape. (b) A plot of the drawing time vs the average affine length of the shape actually drawn by the subject. Affine length was obtained by summing the estimates of affine velocity.

showed no statistically significant variation with the affine transformation or affine length.

Since there were no systematic errors in the reproduction of Euclidean length, we examined the data for systematic deviations in reproducing the presented shape. Errors in reproducing local shape would cause the affine length of the drawn curve to be unequal to the affine length of the presented curve. For example, drawing movements which underestimated the local radius of curvature would result in movements of longer affine length and thus longer total drawing times. To check this we plotted the drawing times vs the cumulative error in the reproduction of local radius of curvature [Fig. 5(a)] as well as vs the affine length of the drawn shapes [Fig. 5(b)]. From Fig. 5 it can be seen that the increase in drawing time was related both to subjects' errors in reproducing the local form and the resulting increase in affine length of the drawn shape.

The results show that the drawing movements, including the errors in reproducing local shape, were performed at constant affine velocity. The deviation from ideal performance of the drawing times provides an illustration of how concepts such as affine length can be useful in the interpretation of drawing movement. It is perhaps useful to speculate upon the source of the error in the drawing movements. One possibility is that the error in reproducing radius of curvature was caused by an inability to match the shape of the presented curve when the radius of curvature was large. Such a conjecture is consistent with previous findings regarding errors obtained during drawing movements. Viviani & Schneider (1991) have reported that drawing movements are more variable for portions of an ellipse with a large radius of curvature than for portions with a small radius of curvature. Moreover, studies exploring cortical mechanisms of the population coding of movement direction, (Georgopoulos *et al.*, 1989) indicated that as the radius of curvature increased past a threshold value, the population coding of movement was no longer predictive of the actual movement (Schwartz, 1994).

CONCLUDING REMARKS

The 1/3 power law of human hand-drawing and planar motion perception has been an intriguing issue since it was first experimentally discovered. In this work we proved theoretically that it is the unique function of curvature that gives a constant affine motion. In other words, two curves which are related by affine transformations are traveled with the same velocity in affine space if, and only if, the velocity is governed by a 1/3 power of the Euclidean curvature. Any other function of curvature will not be affine invariant. Following this, it is not surprising that curves traveled with this power law are perceived (traced) as being covered with constant velocity, since, for example, it is the same velocity with which circle arcs will be traveled. In addition to this theoretical result we presented experimental data which, among other results, illustrated the utility of affine geometry in the analysis of subjects' errors in ideally reproducing a presented shape.

The duality of the 1/3 power law in describing both the production and perception of form suggests that it could originate from a representation common to both visual perception and movement production. If so, our findings indicate that this common representation is best described by affine differential geometry. How and where in the stream of visuomotor processing this representation arises are open questions. While it seems likely that an affine representation would originate from approximations in visuo-motor transformations involving affine rather than Euclidean distances, there is no *a priori* reason to assume that such approximations would be either visual or motor in nature. It has been previously suggested that the duality of the 1/3 power law originates from a motor theory of visual perception (Viviani & Stucchi, 1992). However, an affine perceptual encoding of planar form has certain advantages which suggest that a role of visual representation should not be discounted. For example, affine properties of shape are invariant to relative orientation of the eye and the plane of the drawing motion. Thus, an affine perceptual encoding might simplify the process of drawing the same shape, despite large changes in the relative orientation of the eyes and the hand on the drawing plane. We are currently working on investigations of affine representations for planar motion and their potential role in cortical mechanisms of movement control and perception, as well as further relations between visual and motor tasks.

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