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Large-scale visual frontoparallels under full-cue conditions

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Abstract. We determined the curvature of apparent frontoparallels in a natural scene (a large lawn in broad daylight). Data on frontoparallels in these conditions are very sparse and reveal idiosyncratic curvatures of frontoparallels and irregular variation with distance. We used a method of bisection of linear segments indicated through pairs of stakes at angular separations (from the vantage point) of up to 120 deg. Distances of 2 m and 10 m (in the forward direction) were used. The bisection was carried out by the observer through maneuvering a radio-controlled vehicle carrying a third stake. Four observers participated in the experiment; they had no problems with the task and yielded mutually consistent results. We found that the frontoparallels are significantly curved and are concave towards the observer. Surprisingly, the sign of the curvature is opposite to that found when the frontoparallels are defined through an exocentric pointing task. Available theory (Luneburg's) does not predict this, but the theory is hardly applicable to the case of vision in natural scenes. This interesting discrepancy has not been reported before.

1 Introduction

Optical space (of the human observer) is known to differ appreciably from physical space. Physical space is to an excellent approximation Euclidean. The structure of optical space remains uncertain. A hyperbolic, homogeneous space (Hilbert and Cohn-Vossen 1932/1983) is implied by the perhaps best-known theory, that of Luneburg (1947).

The earliest approaches to the study of optical space have been through the apparent frontoparallels and through the so-called distance and parallel alleys. [An excellent review is in Howard and Rogers (1995).] Conceptually, the frontoparallels are perhaps the simplest. Helmholtz (1867) measured apparent frontoparallels using three vertical threads. When the threads hung in a (physically) frontoparallel plane, he found that the middle one was often seen closer or farther away than the other two, depending on the distance. When he placed beads on the threads, the settings became close to veridical. In experiments like Helmholtz's, one typically reduces all cues except for binocular disparity to the absolute minimum. In addition, the mobility of the observer is severely restricted (no head or body movements); thus the effective field of view is rather limited. Perhaps the most extensive results are those reported by Ogle (1950). The field of view was 32 deg; distances ranged from 20 cm to 6 m. The typical result was that the apparent frontoparallels were convex towards the observer in near space and concave in far space. The change-over point was at about 5 m.

Such results for highly constrained situations in near space are at least qualitatively explained by Luneburg's theory on the basis of an analysis of binocular stereopsis. Here 'explanation' has to be taken *cum grano salis* though, since Luneburg's theory is not a clean-cut analysis of binocular stereo (ie essentially an exercise in geometry), but departs from quite ad hoc prior assumptions. A major assumption made by Luneburg is that visual space is a *homogeneous space*. Empirically that means that spatial configurations may be presented at any location or orientation and still 'look the same' (this 'free mobility' implies the existence of a group of congruences). From a formal perspective this is an extremely strong assumption, since it limits the possibilities to one of the classical geometries: Euclidean, hyperbolic, or elliptic. The assumption of free mobility is quite independent of the setting of binocular vision and might be applied equally well to monocular visual space. We are sceptical in view of the fact that such a theory leaves little room for the systematic variations of space curvature we have encountered earlier (Koenderink et al 2000), for the influence of the actual scene on the structure of the space (visual experience suggests that empty space is different from filled space), or for results that depend upon the task.

Although the results from such reduced-cue situations are of considerable academic interest, they are perhaps less informative for full-cue situations with less-constrained observers. Here 'full cue' may be taken to mean a representative blend of the cues available to visual observers in a natural (outdoors) environment under full daylight. 'Less constrained' means that at least head movements are allowed to scan the scene; even more generously, (in place) body rotations as well. Very little is known about the influence of head movement on stereopsis [for instance see Collewijn et al (1991)], but such freedom of movement is indeed *necessary* when large fields of view (frontoparallels may have extents up to 180 deg) are considered relevant. Notice that the majority of additional cues are 'monocular' ones, ie they are also available to the monocular observer. When such cues can be shown to matter, the data from the purely binocular experiments are thereby rendered inadequate. Moreover, theories like Luneburg's can no longer be considered to apply.

Although the available data on the reduced-cue/constrained-observer situation are extensive, remarkably few systematic data exist for the more natural setting. Of the few studies that qualify, we mention those of Battro et al (1976, 1978), of Wagner (1985), and of ourselves (Koenderink et al 2000, 2001). Unfortunately, the results of these studies fail to add up to a coherent picture.

As a spin-off of experiments with various aims, we have had an opportunity to collect data on apparent frontoparallels. It turns out that the data seem to depend on the precise operationalization. For instance, using an exocentric pointing task we find qualitatively different results from those when we use a line-intersection task. These results are again different from those of the other authors. Apparently the confusion arises owing to differences in details of the operationalization of the frontoparallels.

In order to make progress, it seems desirable to collect systematic data for a number of fairly 'generic' operationalizations. In this paper, we present data on the apparent frontoparallels obtained with the use of an *intersection task*, quite similar to Helmholtz's (1867) original method. In the past, we collected only three-point frontoparallels, essentially collinear triples of two endpoints and a midpoint. Since we are interested in the frontoparallels for very wide fields of view (such as would occur in natural vision) we needed additional (visually) collinear points, much as in the original experiments of Ogle (1950).

Of course, the very notion of 'frontoparallel' becomes somewhat problematic when free eye and head movements are allowed. However, the concerned reader may substitute "straight connection between two targets" for "frontoparallel"—it really makes no difference to the content of this paper. "Frontoparallel" can then be taken to signify "perpendicular to the average viewing direction" which clearly differentiates the geometry from tasks that focus on the radial direction [such as distance (from the observer) bisection tasks] and is a meaningful distinction.

2 Methods

2.1 *General*

Experiments were performed on a large lawn (see figure 1). The grass was somewhat less than ankle-high. The observers had an unobstructed forward range up to about 60 m where a row of bushes and trees obstructed the view. The open range towards

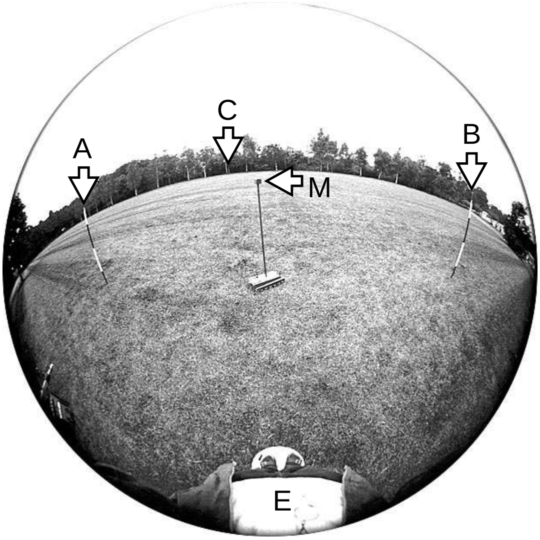

Figure 1. A fish-eye photograph of the experimental situation. The observer ('egocenter') is at E, the feet are placed on the white board. Two stakes, A and B, mark a frontoparallel line segment. A distant stake C defines—together with the egocenter—a visual direction. The task of the observer is to direct the moving vehicle M such that the cube on top of the pole mounted on the vehicle is on the intersection (in the plane through the observer's eyes and the target cube) of the line segment AB with the visual direction EC.

both sides was even wider than that. All sessions were run in broad daylight. The weather is quite variable in the Netherlands, and we experienced the full spectrum between open skies and overcast conditions. Sessions were suspended during rainfall.

The observers were instructed to keep their feet within a 30 cm diameter circle and to maintain normal eye height at all times. Otherwise they were quite free to move (head movements, body rotations). They looked with both eyes. All observers were young adults, naïve to the task and paid for their efforts. The observers were not allowed to see the experimenters change anything in the field or to overhear instructions given by one experimenter to another, nor were they informed about the veridicality of their settings.

The task of an observer was to direct a radio-controlled vehicle towards the intersection (purely by eye-measure, of course) of the line segment AB (say) and a direction EC (say). Here the points A, B, and C are defined through stakes placed at various positions in the field, whereas E denotes the egocenter of the observer. The stake C was always placed at a large distance (large as compared to the locations A, B—50 m in the forward direction) from the observer. Thus the 'line segment' EC is really a *visual direction*. The stakes A and B implicitly define a *line segment*. The stakes were placed at either 2 m or 10 m in the forward direction and various amounts sideways.

The moving vehicle carried a pole on which a cubical target was mounted at 1.5 m above ground level. The observer was instructed to bring the target at the intersection of AB and EC in the horizontal plane at the level of the cube (slightly below the observer's horizon).

Before starting the sessions, each observer was given initial training in driving the radio-controlled vehicle. The observers let the vehicle move to specific targets, turn corners, and drove a number of slaloms. When the sessions were started, the observers could concentrate on their visual task; maneuvering the vehicle did not pose any problems to them.

We monitored the positions of objects in the field (stakes and the cube on the pole on the vehicle) through a geodesic station comprised of a laser rangefinder and a theodolite. Overall, the tolerances are in the millimeter range for an area of about 100 m $\times$ 100 m. The scatter in the data is essentially due to (mainly) the scatter in the settings of the observers and (perhaps partly) the limited accuracy (centimeter range) by which stakes can actually be placed, the measurement accuracy being ample.

2.2 *Specific*

Four observers completed all conditions eight times (except for observer FK who completed all conditions seven times). Results of a fifth observer were discarded because, after completion of the experiments, he confessed to have substituted his own interpretation of the task. (This observer did choose not to look at the target but at the bottom of the vehicle, and he tried to solve the task completely in the ground plane, disregarding the depth dimension altogether.) All in all, we spent a full week running the sessions (five experimenters and the observer), apart from the initial geodesic work in the field (by the crew of experimenters).

We measured apparent frontoparallels at two distances, one quite near (2 m), another at a much larger distance (10 m) from the observer. At each distance, we varied the angular distance between the endpoints of the geodesic arc (30, 60, or 120 deg) and we determined points at various locations along the arc. For the 30 deg separation (stakes at ±15 deg) we measured only the midpoint (0 deg); for the 60 deg separation (stakes at ±30 deg) we measured at −15, 0, and +15 deg; and at the 120 deg separation (stakes at ±60 deg) we measured at −40, −20, 0, +20, and +40 deg. This should yield a representative sampling of the structure of the apparent frontoparallels under full-cue conditions.

All conditions were visited in an overall random order.

3 Experiments and results

The data are shown in figures 2 through 4. We find good agreement for all observers for the near frontoparallels. For the distant frontoparallels, the variability is somewhat larger. Two observers (FS and GV) are almost veridical, but all observers show the same sense of curvatures: in all but few cases the frontoparallels are *concave towards the observer*. This corroborates our (admittedly lacunary as far as frontoparallels go) earlier results (Koenderink et al 2001) with a similar operationalization. The exceptions (opposite sense of curvature) occur for the case of the smallest (30 deg) angle, both in the near and in the far conditions. Here we find occasional convex (towards the observer) arcs, though in these cases the significance of the curvature is low. (Of the five occasions, only one is significant at the 5% level.)

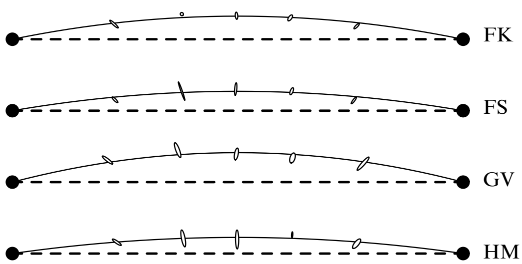

Figure 2. The raw data for all observers (top to bottom: FK, FS, GV, and HM) at the near distance (2 m) and the widest angular extent (120 deg). The data are represented through covariance ellipses; we have fitted a best parabolic arc at these data. Notice that the width between the endpoints is 3.46 m, and the veridical midpoint is at 2 m from the observer.

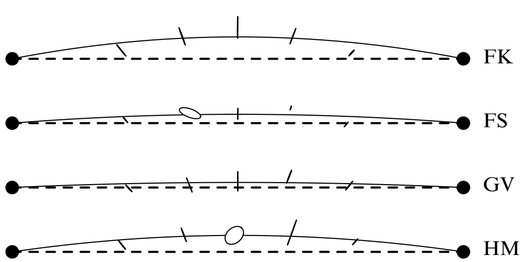

Figure 3. The raw data for all observers (top to bottom: FK, FS, GV, and HM) at the far distance (10 m) and the widest angular extent (120 deg). The data are represented via covariance ellipses; we have fitted a best parabolic arc at these data. Notice that the width between the endpoints is 17.32 m, and the veridical midpoint is at 10 m from the observer.

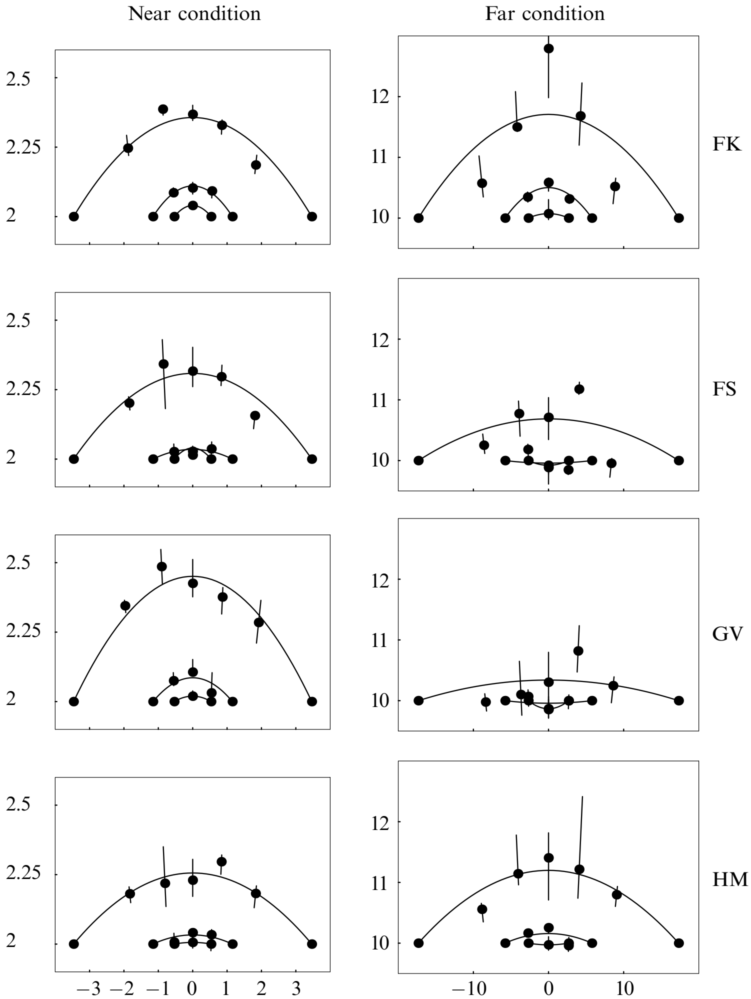

Figure 4. An overview of the full data set plotted in the ground plan. All distances are in meters. The average settings for all angular extents are shown with the standard error in the radial direction.

When we assume that the curved apparent frontoparallels can be well approximated with parabolic arcs (as seems almost self-evident in view of the small curvature and is borne out by the results), we can calculate a curvature value for each individual observation (see appendix). This allows us to perform an ANOVA on the full set of data (curvature as dependent on observer, distance, angular size, and visual direction). An ANOVA on the full data reveals a significant influence of the observer ($F_{3,510} = 12.7$, $p \leqslant 0.0001$), the distance ($F_{1,510} = 155$, $p \leqslant 0.0001$), and the angular extent ($F_{2,510} = 8.9$, $p = 0.0002$), whereas the viewing direction turns out to have no effect ($F_{6,510} = 1.1$, $p > 0.36$). There are three significant interactions, namely observer × distance ($F_{3,510} = 8.9$, $p \leqslant 0.0001$), observer × angular extent ($F_{6,510} = 4.8$, $p \leqslant 0.0001$), and distance × angular extent ($F_{2,510} = 19.7$, $p \leqslant 0.0001$).

In an ANOVA per distance we find a significant influence of the observers ($F_{3,243} > 8.5$, $p \leqslant 0.0001$ for both near and far conditions) and of the angular extent ($F_{2,243} > 9.8$, $p \leqslant 0.0001$ in both the near and far conditions). The interaction observer × angular extent is significant ($F_{6,243} > 3.3$, $p < 0.0035$ in both the near and far conditions).

An ANOVA per distance and per observer reveals a small dependence of the curvature upon the separation. At the near distance, the effect of the angular extent is significant at the 5% level for two observers: observer FK ($F_{2,54} = 35.2$, $p \leqslant 0.0001$) and observer GV ($F_{2,63} = 3.2$, $p = 0.047$). At the far distance, the effect of angular extent is significant at the 5% level for observer FS ($F_{2,63} = 13.7$, $p \leqslant 0.0001$) and observer GV ($F_{2,63} = 5.4$, $p = 0.007$), and not for the others. There is no significant effect of visual direction for any observer at either distance.

The overall curvature (grand total average over all settings per distance) in the near condition is positive for all observers; we find average values ranging from 0.031 ± 0.05 m^{-1} to 0.29 ± 0.14 m^{-1}. For the angular extents of 60 deg and 120 deg, the curvature was significantly different from zero at the 5% level. For the 30 deg separation, the curvature was still significantly different from zero for observers FK and FS.

For the far condition, we find that the curvature for the 120 deg angular separation is significantly different from zero (positive) at the 5% level for all observers. Of course, the curvatures are less well defined for the narrow separations. For this reason, the data for the narrow separations in the far condition are not very informative.

In the near condition, there are some indications that the curvatures may differ with respect to the angular extent. The curvature appears to be somewhat smaller for the narrower separations with respect to the curvature at the 120° separation. This is significant at the 5% level for six out of eight cases.

In the near condition (2 m), the value of the curvature is remarkably similar for all four observers: FK, 0.050 m^{-1}; FS, 0.042 m^{-1}; GV, 0.064 m^{-1}; and HM, 0.036 m^{-1}. In the far condition (10 m), we find values of 0.0093 m^{-1} for FK, 0.0038 m^{-1} for FS, 0.0022 m^{-1} for GV, and 0.0072 m^{-1} for HM. The radius of curvature in the near condition is thus about 21 m (reciprocal of the average curvature over all observers); thus the radius of curvature is quite large with respect to the distance (2 m). At the far distance, the radius of curvature is larger, about 178 m (reciprocal of the average curvature over all observers); thus the radius of curvature is again large with respect to the distance (10 m).

4 Conclusions

We determined the shape of the apparent frontoparallels, roughly in the observer's horizon (plane through the eyes and the target at 1.5 m above ground level) in a full-cue situation with free viewing (head and body rotations) of an observer pinned to a given location. The frontoparallels have been determined by estimating locations on the frontoparallels in specific directions, for fixed endpoints of the frontoparallel arc. Under these conditions, the frontoparallels are curved (radius of curvature about 21 m at 2 m from the observer to about 178 m at 10 m from the observer) with the concave side facing the observer. The concordance between observers is perhaps as good as might have been expected.

Notice that the radius of the Vieth – Müller circle in the near condition is 4 m, in the far condition 20 m. The apparent frontoparallels have radii of curvature that are much larger, about five times in the near and about nine times in the far condition. Thus the geometry in the full-cue situation is quite close to veridical. The curvatures have the same sign as the Vieth—Müller circles though, different from the classical results.

These results do not disagree with the major body of data available from the literature (Battro et al 1976). Unfortunately, Battro et al's data show highly variable results, both with distance for a single observer, and between observers. Our results appear to be far more systematic in character, for reasons that we fail to understand.

The signs of the curvatures are opposite to those measured by us (Koenderink et al 2000) with a (completely) different task, namely an exocentric pointing task (especially

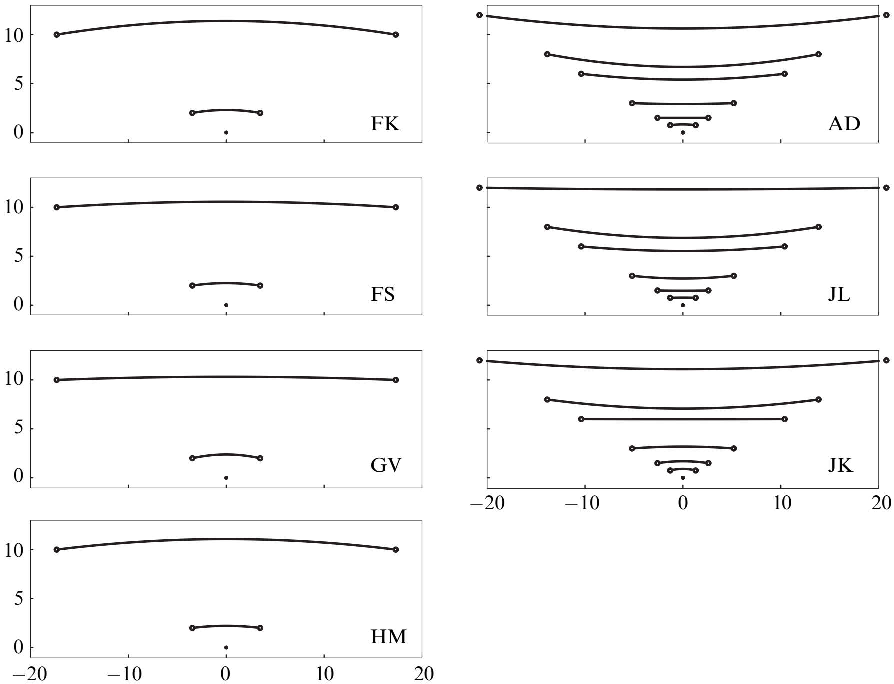

Figure 5. The 'apparent frontoparallels' as determined by the bisection task for four observers (left), and the 'apparent frontoparallels' as determined by an exocentric pointing task for three (different) observers (right). All measures are in meters.

for the large distance, see figure 5). This is perhaps a puzzling result. It suggests that the two tasks reveal different aspects of visual space. After all, 'apparent frontoparallel' is only a name for an operationally defined entity; thus the findings are not necessarily in conflict. A task dependence seems likely, for otherwise the conditions in the two experiments were quite similar.

For either task it is easy enough to think of 'reasons' that would 'account for' deviations. For instance, in the case of the pointing task, one might assume that the orientation of the pointer was systematically misperceived; in the case of the bisection, that distance was systematically misperceived; etc. No doubt, one could come up with simple models that would 'explain' the differences. However, we consider such attempts to be inadmissible in the context of 'visual space'. For instance, the pointing device itself is in visual space and its 'misperception' should find its explanation in terms of its being in visual space. Thus, an explanation along these lines calls for a very intricate analysis which had better be avoided in view of the fact that the structure of visual space is still insufficiently understood. The alternative would be to discard the notion of 'visual space' altogether. We consider this an entirely reasonable direction to explore, and perhaps in the long run the only viable option. The drawback of such a move is that the various sets of data become completely uncoupled: the pointing results and the present results then have to be explained separately and don't bear upon each other. The notion of 'visual space' is so valuable exactly because it yields a *general framework* in which the results from different tasks can be integrated. One should not fail to explore how far the notion of 'visual space' (suitably amended as more empirical data become available) can be stretched. We have the feeling that we are presently at the fringes of the usefulness of the notion, but it is still too early to discard it offhand.

The possibility that the curvatures may differ with respect to the angular extent might be interpreted as an indication that the curvature of the apparent frontoparallels varies along their length (thus they would deviate from circular arcs). The evidence is rather weak though, in view of the fact that the narrower arcs are less well defined and are more prone to systematic errors than the large ones. Another interpretation might be that perhaps the concept of 'apparent frontoparallel' itself is incoherent. This is empirically decidable, at least in principle. Suppose two apparent frontoparallels both contain two disjunct points $\mathcal{A}$ and $\mathcal{B}$ (say). Then, when one of these frontoparallels contains any point $\mathcal{P}$ (say) that fails to lie on the other, one already has an inconsistency. This is evidently up to empirical verification. The possibility is an exciting one, for such a state of affairs would kill the very notion of a 'visual space'. On the basis of the present evidence we deem it prudent to defer judgment though.

One might argue that the main 'monocular cue' pertinent to the task is not a depth cue at all, but rather the projective structure of the visual field. For it would be possible to perform the task perfectly by drawing straight lines on the projection of the ground plane (this could be done on a photograph taken from the vantage point of our observers, say) through the points where the stakes (or the pole on the vehicle) met the ground plane. In this case perhaps 'subjective curvatures' in the visual field might be expected to play a role [for instance, see Hauck (1875) and Pirenne (1970)]. This isn't too strong an argument though, since the subjective curvature disappears when one fixates a point on the line. The results suggest that our observers did *not* try to solve the task this way.

The instructions were to do the geometrical task on the position of the cube on the vehicle, thus 1.5 m above ground level. Yet it is certainly true that the observers looked around constantly and were very much aware of the ground plane itself. It is indeed *necessary* that the observers see the ground plane, otherwise we would be back in a very unnatural, reduced-cue situation. For instance, the points could have been defined through balloons (of random diameters) on thin (invisible) threads. In such a case the distances to the points would become very uncertain, since the observers would have to rely on mere binocular cues. The ground plane *has* to be visible, despite the methodological drawback that this—at least in principle—allows the task to be done in the visual field, rather than the visual world. Although this is a problem in principle, we don't believe it to be one in practice. Debriefing of the observers at the conclusion of all sessions suggests that all observers thought of the task as essentially having to do with distance judgments.

What is perhaps most important is that all observers are very reliable (the scatter in repeated settings is small) in judging rectilinearity in the frontoparallel for visual angles up to 120 deg. In such cases the observers almost certainly have to integrate a number of distinct glances, sampled at different eye and head positions. Apparently this poses few problems to the human (stationary) observer.

These results are quantitatively, but even qualitatively, different from the classical results obtained under reduced-cue conditions (binocular disparity the only effective cue) and reduced viewing conditions (fixed skull). Of course, the classical results are only available up to a few meters and for angular subtends of less than 30 deg. Our results apply to a much wider range. But the sign of the curvature of the apparent frontoparallels is clearly at odds for the two conditions. Apparently the 'monocular cues' change the geometry of visual space decisively.

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APPENDIX

The essential geometry is indicated in figure A1. The observer has to put a point ($\mathcal{D}$) on the apparent frontoparallel that connects the points (stakes in the field) $\mathcal{P}$ and $\mathcal{Q}$. The egocentric direction toward the point is indicated via a distant stake.

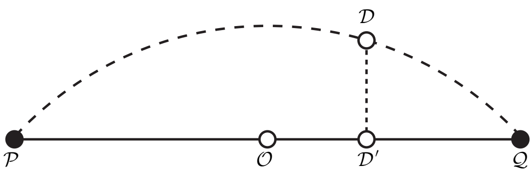

Figure A1. The observer has to put a point ($\mathcal{D}$) on the apparent frontoparallel that connects the points (stakes in the field) $\mathcal{P}$ and $\mathcal{Q}$. The point has to be put at horizontal distance $x_{\mathcal{D}}$ (say) from the center point $\mathcal{O}$. (Of course, $x_{\mathcal{D}}$ is indicated as an egocentric direction via a distant stake.) Let the distance $\mathcal{PQ}$ be denoted L. The point indicated by the observer can be characterized via the (signed) distance $y_{\mathcal{D}} = \mathcal{DD}'$.

Let the perpendicular from the point $\mathcal{D}$ to the objectively frontoparallel segment $\mathcal{PQ}$ be denoted $\mathcal{D}'$. It is convenient to parameterize point $\mathcal{D}$ via the signed distances $x_{\mathcal{D}} = \mathcal{OD}'$ ($\mathcal{O}$ the midpoint of the segment $\mathcal{PQ}$) and $y_{\mathcal{D}} = \mathcal{DD}'$. The parameters $x_{\mathcal{D}}$ and $y_{\mathcal{D}}$ are simply related to the geometry of the setup.

Let the distance $\mathcal{PQ}$ be denoted L. Parabolic arcs that connect the endpoints $\mathcal{P}$ and $\mathcal{Q}$ satisfy the general expression

$$y(x) = \tfrac{1}{2}\kappa(x - \tfrac{1}{2}L)(x + \tfrac{1}{2}L)\,,$$

for, clearly, $y = 0$ at the endpoints (where $x = \pm\frac{1}{2}L$) and the expression is quadratic in x. These parabolic arcs are thus characterized by a single parameter κ, which denotes their *curvature*.

When you substitute the parameters of a data point $\{x_{\mathcal{D}}, y_{\mathcal{D}}\}$ into this expression and solve for the curvature you obtain

$$\kappa_{\mathcal{D}} = \frac{2y_{\mathcal{D}}}{(x_{\mathcal{D}} - \frac{1}{2}L)(x_{\mathcal{D}} + \frac{1}{2}L)}\,.$$

Thus each individual data point $\{x_{\mathcal{D}}, y_{\mathcal{D}}\}$ yields an estimate of the curvature $\kappa_{\mathcal{D}}$.

In the analysis, we have converted all individual data points to their equivalent curvatures. This allows us to perform an ANOVA on the full data.