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Nonparametric measures of angular-angular association

By N. I. FISHER
C.S.I.R.O. Division of Mathematics and Statistics, Sydney, Australia

AND A. J. LEE
Department of Mathematics, University of Auckland, Auckland, New Zealand

Summary

A general model is proposed for association between two angular variables, corresponding to monotone association between two linear variables. A U-statistic analogous to Kendall's tau is developed to estimate the degree of this association and its distributional properties studied. A simple modification of Mardia's rank angular-angular correlation coefficient is proposed as the appropriate analogue of Spearman's rho for assessing this association.

Some key words: Directional data; Rank correlation; Toroidal concordance; U-statistic.

1. Introduction

Let Θ and Φ be two angular random variables, with joint distribution concentrated on the surface of a torus, and let $(\theta_1, \phi_1), \ldots, (\theta_n, \phi_n)$ be n independent realizations of (Θ, Φ). The problem of testing for general association between Θ and Φ has received little attention in the literature and, to date, no general model for such association akin to a monotone relationship between two real random variables has been proposed.

By adapting the test proposed by Hoeffding (1948b) of the independence of two real continuous random variables, Rothman (1971) obtained a test consistent against all alternative continuous bivariate angular models. He obtained the asymptotic distribution of the test statistic, but not its small-sample distribution. Hillman (1974) suggested a method requiring computation of the maximum and minimum values of the n^2 Spearman's rank correlations obtained by choosing all possible positions for the reference directions of the θ's and ϕ's. Hillman tabulated the statistic for $n = 5, \ldots, 11$, but the large-sample distribution is unknown. Mardia (1975) suggested that the appropriate analogue, for dependence of Θ and Φ, of a linear relationship between two real random variables, is

$$l\Theta \pm m\Phi + \psi_0 \equiv 0 \pmod{2\pi},$$

where l and m are positive integers and ψ_0 is an unknown constant angle. For the particular case in which l and m can be assumed to be one, Mardia developed an analogue of Spearman's rank correlation coefficient as follows. Denote by $r_1, \ldots, r_n$ and $s_1, \ldots, s_n$ the separate sample linear ranks of $\theta_1, \ldots, \theta_n$ and $\phi_1, \ldots, \phi_n$ respectively, and set

$$n^2 \bar{R}_1^2 = [\Sigma \cos\{2\pi(r_i - s_i)/n\}]^2 + [\Sigma \sin\{2\pi(r_i - s_i)/n\}]^2,$$
$$n^2 \bar{R}_2^2 = [\Sigma \cos\{2\pi(r_i + s_i)/n\}]^2 + [\Sigma \sin\{2\pi(r_i + s_i)/n\}]^2.$$

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Then $r_0 = \max(\bar{R}_1^2, \bar{R}_2^2)$ is a measure of the degree of dependence of the form

$$\theta+\phi+\psi_0 \equiv 0 \pmod{2\pi} \tag{1·1}$$

or

$$\theta-\phi+\psi_0 \equiv 0 \pmod{2\pi} \tag{1·2}$$

and is invariant under choice of reference direction for Θ or Φ, or equivalently, the value of ψ_0. Clearly, $0 \leqslant r_0 \leqslant 1$, with $r_0 = 1$ corresponding either to $\bar{R}_1^2 = 1$ and complete 'positive' dependence as in (1·1), or to $\bar{R}_2^2 = 1$ and complete 'negative' dependence as in (1·2). Mardia provides selected percentiles for the small-sample distributions, and shows that $2(n-1)\bar{R}_1^2$ and $2(n-1)\bar{R}_2^2$ are asymptotically distributed as independent χ_2^2 variates, so that the asymptotic distribution of $2(n-1)r_0$ is known.

In this paper, simple generalizations of (1·1) and (1·2) are proposed as natural models of general association between two angular random variables. The extent of this association can be estimated using a U-statistic analogous to Kendall's tau. In §3, the small-sample and large-sample properties of this statistic are obtained, and in §4 some examples given of its application. In §5·1, some comments are made on these models *vis-à-vis* more general models. In §5·2, a slight modification of Mardia's r_0 statistic is suggested as the Spearman's rho-type statistic corresponding to the Kendall's tau-type statistic considered herein, and its distribution tabulated.

The discussion parallels that given in an earlier paper (Fisher & Lee, 1981) on general association of an angular and a linear random variable; comments on applications of the methods to problems of partial association can be found there.

2. A GENERAL MODEL FOR DEPENDENCE OF TWO ANGULAR RANDOM VARIABLES

Corresponding to the approach in Fisher & Lee (1981) we postulate that a reasonable model for general association between Θ and Φ is any relationship $\phi = g(\theta)$ such that, as θ moves continuously through a complete revolution in a particular sense, either clockwise or anticlockwise, ϕ also moves continuously in a particular sense through a complete revolution. If the senses are the same, the relationship $\phi = g(\theta)$ is said to be toroidally-concordant, or T-concordant, otherwise if one sense is clockwise and the other counter-clockwise, $\phi = g(\theta)$ is T-discordant. Thus $\phi = g(\theta)$ is essentially a warped version of (1·1) or (1·2) according as the relationship is T-concordant or T-discordant.

Now let $p_i = (\theta_i, \phi_i)$ $(i = 1, \ldots, n)$ be n points on the torus. Define the points to be T-concordant if there exists a T-concordant relationship $\phi = g(\theta)$ such that $\phi_i = g(\theta_i)$ $(i = 1, \ldots, n)$. The points are defined to be discordant in a corresponding way. Finally, let $P = (\Theta, \Phi)$ be randomly distributed on the torus. We propose to measure association between Θ and Φ by

$$\Delta = \text{pr}(P_1, P_2, P_3 \text{ are } T\text{-concordant}) - \text{pr}(P_1, P_2, P_3 \text{ are } T\text{-discordant}),$$

where P_1, P_2 and P_3 are independent random points distributed as P.

Note that the notion of T-concordance is independent of the choice of reference direction for Θ or Φ, and Δ inherits this property. Any three points on a torus are either T-concordant or T-discordant.

To obtain a U-statistic estimate of Δ, define the kernel

$$\delta(p_1, p_2, p_3) = \begin{cases} 1 & \text{if } p_1, p_2, p_3 \text{ are } T\text{-concordant}, \\ -1 & \text{if } p_1, p_2, p_3 \text{ are } T\text{-discordant}. \end{cases}$$

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As with Kendall's tau, δ depends only on the separate-sample ranks of $\theta_1, \theta_2, \theta_3$ and ϕ_1, ϕ_2, ϕ_3, and is most efficiently calculated from the representation

$$\delta(p_1, p_2, p_3) = \operatorname{sgn}(\theta_1 - \theta_2) \operatorname{sgn}(\theta_2 - \theta_3) \operatorname{sgn}(\theta_3 - \theta_1) \times \operatorname{sgn}(\phi_1 - \phi_2) \operatorname{sgn}(\phi_2 - \phi_3) \operatorname{sgn}(\phi_3 - \phi_1). \quad (2\cdot1)$$

Then the U-statistic for estimating Δ is

$$\hat{\Delta}_n = \binom{n}{3}^{-1} \sum_{1 \leqslant i < j < k \leqslant n} \delta(p_i, p_j, p_k),$$

and $\hat{\Delta}_n$ has the following properties:

(i) $-1 \leqslant \Delta_n \leqslant 1$;

(ii) $\hat{\Delta}_n = +1$ if $p_1, \ldots, p_n$ are T-concordant and -1 if they are T-discordant;

(iii) if Θ and Φ are independent, and $P_1, \ldots, P_n$ are distributed independently as P, then $\hat{\Delta}_n\,(P_1, \ldots, P_n)$ is distributed symmetrically about 0.

If there are tied observations amongst $\theta_1, \ldots, \theta_n$ or $\phi_1, \ldots, \phi_n$, any kernel value $\delta(p_1, p_2, p_3)$ containing a tied pair can be set to zero, and the combinatorial divisor in the definition of $\hat{\Delta}_n$ should be reduced by one for each such zero value.

3. Distribution theory for $\hat{\Delta}_n$

The method of obtaining the sampling properties of $\hat{\Delta}_n$ closely parallels that given in §3 of Fisher & Lee (1981). To derive these properties under the null hypothesis that Θ and Φ are independent, let $P_i = (\Theta_i, \Phi_i)$ $(i = 1, 2, 3)$ be independently distributed as (Θ, Φ), and suppose that Θ and Φ are independent variates on the circle.

Let $P = (\theta, \phi)$ be a point on the torus, with some fixed origin for each coordinate. Let $P^* = (\theta^*, \phi^*)$ be the point $(\frac{1}{2}\theta/\pi, \frac{1}{2}\phi/\pi)$ corresponding to P on the unit square. The transformation $P \rightarrow P^*$ depends of course on the choice of origins on the torus.

Define $\delta^*(P_1^*, P_2^*, P_3^*) = \delta(P_1, P_2, P_3)$; then δ^* can be calculated by (2·1) and depends only on the ranks of θ_i^* and ϕ_i^*, and does not depend on the choice of origin.

Thus the distribution of $\delta(P_1, P_2, P_3) = \delta^*(P_1^*, P_2^*, P_3^*)$ is invariant under monotone transformations of P_i^*, and so we may assume that the P_i^* are uniformly distributed on the unit square. Dropping the * notation and working on the unit square, we define

$$\delta_2(p_1, p_2) = E\{\delta(p_1, p_2, P_3)\}, \quad \sigma_2^2 = \operatorname{var}\{\delta_2(P_1, P_2)\},$$

$$\delta_1(p_1) = E\{\delta_2(p_1, P_2)\}, \quad \sigma_1^2 = \operatorname{var}\{\delta_1(P_1)\},$$

so that $E\{\delta_1(P_1)\} = E\{\delta_2(P_1, P_2)\} = 0$. A routine calculation yields

$$\delta_2(p_1, p_2) = f(\theta_1 - \theta_2)\, f(\phi_1 - \phi_2),$$

where $f(x) = \operatorname{sgn}(x)(1 - 2|x|)$. Note that

$$\int_0^1 \int_0^1 f(x-y)\, dx\, dy = 0$$

and that

$$\delta_1(p_1) = E\{\delta_2(p_1, P_2)\} = \int_0^1 f(\theta_1 - \theta_2)\, d\theta_2 \int_0^1 f(\phi_1 - \phi_2)\, d\phi_2 = 0:$$

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further,

$$\sigma_1^2 = \text{var}\{\delta_1(P_1)\} = 0, \quad \sigma_2^2 = \text{var}\{\delta_2(P_1, P_2)\} = \tfrac{1}{9}, \quad \sigma_3^2 = \text{var}\{\delta(P_1, P_2, P_3)\} = 1.$$

Thus (Hoeffding, 1948a) $\hat{\Delta}_n$ is degenerate of order one, and

$$\text{var}(\hat{\Delta}_n) = \binom{n}{3}^{-1} \sum_{c=1}^{3} \binom{3}{c}\binom{n-3}{3-c}\sigma_c^2 = \frac{2(n-3)+6}{n(n-1)(n-2)}.$$

Using standard methods (Gregory, 1977; Sproule, 1974) the asymptotic distribution of $n\hat{\Delta}_n$ is found to be identical to that of $3\,\Sigma_\nu \lambda_\nu(Z_\nu^2 - 1)$, where the sum is over $\nu = 1, \ldots, \infty$, where Z_ν are independent normal $N(0,1)$ random variables, and the λ_ν are the eigenvalues of the integral equation with kernel $f(\theta_1 - \theta_2)\, f(\phi_1 - \phi_2)$. These eigenvalues are of the form $\lambda_\nu = (\pm\mu_l)(\pm\mu_m)$ $(l, m = 1, 2, \ldots)$, where $\pm\mu$ are the eigenvalues of the integral equation with kernel $f(\theta_1 - \theta_2)$, so that $\mu_l = i/(\pi l)$.

Thus the asymptotic distribution of $n\Delta_n$ is that of

$$\frac{3}{\pi^2} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{lm} W_{lm}, \tag{3·1}$$

where the W_{lm} are independently distributed as the difference of two independent χ_2^2 variates.

The asymptotic distribution of $n\hat{\Delta}_n$ was computed by approximating the characteristic function of (3·1) by that of

$$\frac{3}{\pi^2} \sum_{lm \leqslant N} \frac{1}{lm} W_{lm} + cZ, \tag{3·2}$$

where c is chosen to make the variance of (3·1) and (3·2) coincide and Z is a $N(0, 1)$ random variable. The distribution function was then computed by numerical inversion.

It is feasible to enumerate the complete distribution for $n = 3, \ldots, 7$ by generating all $n!$ permutations $(\pi_1, \ldots, \pi_n)$ of $(1, \ldots, n)$ and computing $n\hat{\Delta}_n$ for each of the $n!$ sets $\{(i, \pi_i);\, i = 1, \ldots, n\}$. These distributions are tabulated in Table 1. For $n > 7$, 5000 random permutations were used to estimate the distribution for each sample size considered. Selected percentiles of these distributions are given in Table 2; beyond $n = 30$, the asymptotic distribution furnishes an adequate approximation. Tests of the hypothesis $\Delta = 0$ against the alternative $|\Delta| = 1$ are then carried out by referring $|n\hat{\Delta}_n|$ to Table 1 and rejecting the hypothesis if $|n\hat{\Delta}_n|$ is too large; similarly one-sided tests can be performed.

TABLE 1. *Probability function of* $n\hat{\Delta}_n$ $(n = 3, 4, 5, 6, 7)$; *distributions are symmetric about* 0

$n = 3$	x	3	−3										
	$2\,\text{pr}(3\hat{\Delta}_3 = x)$	1	1										
$n = 4$	x	4	0	−4									
	$6\,\text{pr}(4\hat{\Delta}_4 = x)$	1	4	1									
$n = 5$	x	5	2	1	0	…							
	$24\,\text{pr}(5\hat{\Delta}_5 = x)$	1	5	5	2								
$n = 6$	x	6·0	3·6	2·4	1·2	0·0	…						
	$120\,\text{pr}(6\hat{\Delta}_6 = x)$	1	6	12	23	36							
$n = 7$	x	7·0	5·0	3·8	3·4	2·6	2·2	1·8	1·4	1·0	0·6	0·2	…
	$720\,\text{pr}(7\hat{\Delta}_7 = x)$	1	7	14	21	14	21	63	44	28	70	77	

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TABLE 2. *Selected percentiles* $x_{n,\alpha}$ *of the distribution of* $n\hat{\Delta}_n$; $\mathrm{pr}(n\hat{\Delta}_n \leqslant x_{n,\alpha}) = 1-\alpha$, $\mathrm{pr}(|n\hat{\Delta}_n| \leqslant x_{n,\alpha}) = 1-2\alpha$; α, *tail probability*; n, *sample size*

n	$\alpha = 0{\cdot}001$	$\alpha = 0{\cdot}005$	$\alpha = 0{\cdot}01$	$\alpha = 0{\cdot}025$	$\alpha = 0{\cdot}05$	$\alpha = 0{\cdot}1$
8	6·23	4·74	4·25	3·40	2·78	2·10
9	6·08	4·64	4·16	3·33	2·72	2·07
10	5·96	4·56	4·09	3·28	2·68	2·04
11	5·85	4·50	4·02	3·24	2·65	2·01
12	5·77	4·44	3·97	3·20	2·62	1·99
13	5·70	4·40	3·93	3·17	2·59	1·97
14	5·64	4·36	3·89	3·14	2·57	1·96
15	5·59	4·33	3·86	3·12	2·56	1·95
20	5·40	4·21	3·75	3·04	2·49	1·90
25	5·29	4·14	3·68	2·99	2·46	1·87
30	5·22	4·09	3·64	2·96	2·43	1·86
∞	4·85	3·85	3·42	2·81	2·31	1·77

If Θ and Φ are not independent, $\sigma_1^2 \neq 0$ in general, so the asymptotic distribution of $n^{\frac{1}{2}}(\Delta_n - \Delta)$ will be normal with mean zero and variance $9\sigma_1^2$. It follows that an asymptotic confidence interval for Δ is $\hat{\Delta}_n \pm 3\hat{\sigma}_{1,n} z(\frac{1}{2}\alpha)/\sqrt{n}$, where $z(\frac{1}{2}\alpha)$ is the upper $100(1-\frac{1}{2}\alpha)$-percentage point of the $N(0,1)$ distribution and $\hat{\sigma}_{1,n}^2$ is a consistent estimate of σ_1^2 (Puri & Sen, 1971, p. 58) given by

$$\hat{\sigma}_{1,n}^2 = \frac{1}{n-1}\sum_{l=1}^{n} (\hat{\Delta}_n^{(l)} - \hat{\Delta}_n)^2,$$

where

$$\hat{\Delta}_n^{(l)} = \binom{n-1}{2}^{-1} \sum_{1 \leqslant i < j \leqslant n} \delta(P_l, P_i, P_j) \quad (i, j \neq l).$$

4. EXAMPLES

We now apply the techniques described in the previous sections to estimate angular-angular association in 2 sets of data.

Example 1 (Downs, 1974). The peak times for two successive measurements of blood pressure, converted into angles, of 10 medical students were recorded. The estimated association between the two sets of readings is $\hat{\Delta}_{10} = 0{\cdot}6333$, which is significantly different from zero at the 1% level. An approximate 95% confidence interval for Δ is (0·530, 0·737).

Example 2 (Johnson & Wehrly, 1977). Wind directions were measured at 6 a.m. and 12 noon on each of 21 consecutive days, at a weather station in Milwaukee. The estimated association between the 6 a.m. and 12 noon measurement is $\hat{\Delta}_{21} = 0{\cdot}2140$ which is significant at the 5% level. An approximate 95% confidence interval for Δ is (0·170, 0·258).

5. DISCUSSION

5·1. *More general models*

We have only generalized the simplest form of the model $l\Theta \pm m\Phi + \psi_0 \equiv 0 \pmod{2\pi}$ to obtain the notion of toroidal concordance. The statistic $\hat{\Delta}_n$ will presumably have some

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worth even if l or m is greater than 1, and unknown. In this latter case, however, one should not expect to obtain a particularly good test of independence of Θ and Φ, since the variables measured will be $\Theta_l = l\Theta \pmod{2\pi}$ and $\Phi_m = m\Phi \pmod{2\pi}$ from which the original Θ and Φ cannot be recovered. There is some comment on this problem in the discussion of Mardia (1975).

5·2. *An analogue of Spearman's rho for assessing T-concordance*

In view of the existence of the notion of positive and negative association, it seems sensible to adapt Mardia's r_0 statistic to the interval $[-1, 1]$ by defining $\hat{\Pi}_n = \bar{R}_1^2 - \bar{R}_2^2$, and so obtaining an appropriate analogue of Spearman's rho. Tables 3 and 4 contain the distribution of $(n-1)\hat{\Pi}_n$ for $n = 3, \ldots, 7$, and selected percentiles for larger n, respectively. Because $2(n-1)\bar{R}_1^2$ and $2(n-1)\bar{R}_2^2$ are, asymptotically, distributed independently as χ_2^2-variates, the asymptotic distribution of $(n-1)\hat{\Pi}_n$ is double exponential with density $\frac{1}{2}e^{-|x|}$ $(-\infty < x < \infty)$; some percentiles are also given in Table 4.

TABLE 3. *Probability function of* $(n-1)\hat{\Pi}_n$ $(n = 3, 4, 5, 6, 7)$; *distributions are symmetric about* 0

$n = 3$	x	2	−2									
	$2\,\mathrm{pr}(2\hat{\Pi}_3 = x)$	1	1									
$n = 4$	x	3	0	−3								
	$6\,\mathrm{pr}(3\hat{\Pi}_4 = x)$	1	4	1								
$n = 5$	x	4	1·79	0	...							
	$24\,\mathrm{pr}(4\hat{\Pi}_5 = x)$	1	5	12								
$n = 6$	$12x$	60	40	25	20	15	5	0	...			
	$120\,\mathrm{pr}(5\hat{\Pi}_6 = x)$	1	6	6	3	14	12	36				
$n = 7$	x	6	4·71	3·68	3·47	2·78	2·71	1·93	1·83	1·81	1·54	1·33
	$720\,\mathrm{pr}(6\hat{\Pi}_7 = x)$	1	7	7	7	14	7	28	7	7	28	14
$n = 7$	x	1·07	0·86	0·69	0·59	0·52	0·48	0·38	0·31	0·71	0	
	$720\,\mathrm{pr}(6\hat{\Pi}_7 = x)$	28	42	28	7	7	28	28	7	14	88	

Table 4. *Selected percentiles* $y_{n,\alpha}$ *of the distribution of* $(n-1)\hat{\Pi}_n$; $\mathrm{pr}\,(n-1)\hat{\Pi}_n \leqslant y_{n,\alpha} = 1-\alpha$, $\mathrm{pr}\,|(n-1)\hat{\Pi}_n| \leqslant y_{n,\alpha} = 1-2\alpha$; α, *tail probability;* n, *sample size*

n	$\alpha = 0{\cdot}005$	$\alpha = 0{\cdot}01$	$\alpha = 0{\cdot}025$	$\alpha = 0{\cdot}05$	$\alpha = 0{\cdot}10$
8	4·80	4·11	3·26	2·55	1·80
9	4·78	4·09	3·23	2·52	1·78
10	4·77	4·07	3·21	2·50	1·76
11	4·75	4·06	3·19	2·48	1·74
12	4·74	4·04	3·17	2·46	1·73
13	4·73	4·03	3·15	2·45	1·72
14	4·72	4·02	3·14	2·44	1·71
15	4·71	4·02	3·13	2·43	1·71
20	4·67	3·99	3·10	2·40	1·68
25	4·64	3·97	3·07	2·38	1·67
30	4·63	3·96	3·06	2·36	1·66
∞	4·60	3·91	2·99	2·30	1·61

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For random samples of bivariate data in the plane, Spearman's rho is just the projection of Kendall's tau into the space of linear rank statistics (Hájek & Sĭdák, 1967, pp. 60–1). It is interesting to speculate that some corresponding result may be true for $\hat{\Pi}_n$, a quadratic rank statistic, and $\hat{\Delta}_n$.

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