Biol. Cybern. 82, 69–83 (2000)

Biological
Cybernetics

Haptic interaction with virtual objects

Spatial perception and motor control

Ernest D. Fasse[1,*], **Neville Hogan**[1,2], **Bruce A. Kay**[2,**], **Ferdinando A. Mussa-Ivaldi**[2,***]

[1] Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
[2] Department of Brain and Cognitive Sciences, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Received: 12 February 1996 / Accepted in revised form: 6 May 1999

Abstract. This paper considers interaction of the human arm with "virtual" objects simulated mechanically by a planar robot. Haptic perception of spatial properties of objects is distorted. It is reasonable to expect that it may be distorted in a geometrically consistent way. Three experiments were performed to quantify perceptual distortion of length, angle and orientation. We found that spatial perception is geometrically inconsistent across these perceptual tasks. Given that spatial perception is distorted, it is plausible that motor behavior may be distorted in a way consistent with perceptual distortion. In a fourth experiment, subjects were asked to draw circles. The results were geometrically inconsistent with those of the length perception experiment. Interestingly, although the results were inconsistent (statistically different), this difference was not strong (the relative distortion between the observed distributions was small). Some computational implications of this research for haptic perception and motor planning are discussed.

1 Introduction

Haptic spatial perception is distorted. When asked to make judgments about spatial properties of objects such as length, angle and orientation by feeling them, subjects make systematic errors. This paper reports an investigation of haptic interaction of the human arm with "virtual" objects. Our goal was to understand more about the computational processes presumed to underlie sensory-motor behavior. Objects were simulated using a robotic manipulandum (shown schematically in Fig. 11) which restricted movement to a horizontal plane. Interaction forces were generated by the manipulandum's actuators in response to commands computed at each sampling instant based on measurements of position and velocity and a model of the object to be simulated.

1.1 Geometric structure

We assumed that interaction of the human with the manipulandum over an interval of time is described completely by a configuration and force trajectory, $[x(t), f(t)]$. Such a trajectory shall be referred to as an (idealized) dynamic stimulus. The projection of a configuration and force trajectory onto the configuration trajectory only, $x(t)$, shall be referred to as a spatiotemporal stimulus. The underlying path, $x()$, which is independent of the instantaneous velocities of the path (independent of the temporal parameterization), shall be referred to as the spatial stimulus.

During interaction, afferent and efferent information is acquired. Afferent information comes from mechanoreceptors such as cutaneous and deep tissue sensors, muscle spindles, Golgi tendon organs and joint capsule receptors. Efferent information is available from so-called corollary discharge, information available from motor areas of the central nervous system (CNS) that project onto sensory areas. Percepts of arm state and of the simulated objects are formed based on afferent and efferent information acquired during interaction, and on prior knowledge (e.g., that the simulated object is a rectangle and not an arbitrary polygon). For simplicity, we assumed that afferent information was a function of dynamic stimulus only, and that efferent information was a function of motor intent only.

Spatial perception can be viewed as an integrative, computational process in which spatial properties are inferred from instantaneously acquired efferent and/or afferent information, and prior knowledge. Spatial properties of objects, such as length of segments, continuity of paths, angles between surfaces, etc., can be

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Correspondence to: E.D. Fasse

Present addresses:
*Department of Aerospace and Mechanical Engineering, Tucson, AZ 85721-0119, USA
**Department of Cognitive and Linguistic Sciences, Campus Box 1978, Brown University, Providence, RI 02912, USA
***Department of Physiology M211, Northwestern University, Medical School, 303 East Chicago Avenue, Chicago, IL 60611, USA

theoretically determined based on the spatial stimulus only. It is therefore useful to think of the perceptual processes as implementing an underlying, abstract geometrical reasoning system. This led us to several interesting questions that we addressed experimentally: (1) what is the structure of the abstract geometrical reasoning system implemented by the computational processes? (2) are the processes mutually consistent with respect to the underlying system?

In our investigation, we considered the question of to what degree there is a perceived metric of space, i.e., to what degree humans are capable of perceiving the distance traversed along paths. We assumed initially that the perceived distance depends only on the path (the spatial stimulus). This need not be the case, of course. The perceived distance might depend on the direction or speed at which the path was traversed. It might depend on whether or not the path resulted from an unimpeded movement or from interaction with an object. If the path resulted from interaction with an object, the perceived distance might depend on the texture of the object's surface, how hard the subject pressed against the surface, or prior knowledge about the object's shape. Although these factors and others no doubt significantly influence perception of distance, we expected that the path would be the most important factor and focused our experiments on it.

We hypothesized that human behavior is metrically consistent. Consider a subject interacting with a simulated object: we hypothesized that the perceived extent of the object depends only on the spatial stimulus (the path traversed by the hand). An idealized subject's distorted but metrically consistent perception of a spatial stimulus is a local, linear stretch of the stimulus. This is illustrated in Fig. 1.

This leads to testable predictions about human behavior. One of the more interesting predictions is that the perception of length and angle is expected to be related. For example, if a certain rectangle is perceived to be square, then if the rectangle is cut in half along the diagonal, one would expect the acute angles of the resulting right triangle to be perceived as equal. In the experiments described below we found that haptic spatial perception is not geometrically consistent; specifically, we found that perception of an angle was distorted, but in a way that could not be predicted by knowing how perception of length was distorted. This is surprising given that, as Foley (1972) observed: "If an observer's words are any index of his thoughts, then man (or at least the M.I.T. student!) is cognitively a Euclidean. In thinking about geometrical problems, he tends to make Euclidean assumptions." However, our results show that human haptic perception cannot be described as Euclidean.

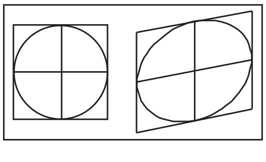

Fig. 1. An idealized observer's distorted but metrically consistent perception can be thought of as a linear stretch of the stimulus. The observer's (distorted) perception of the shape on the *left* is identical to an undistorted perception of the shape on the *right*

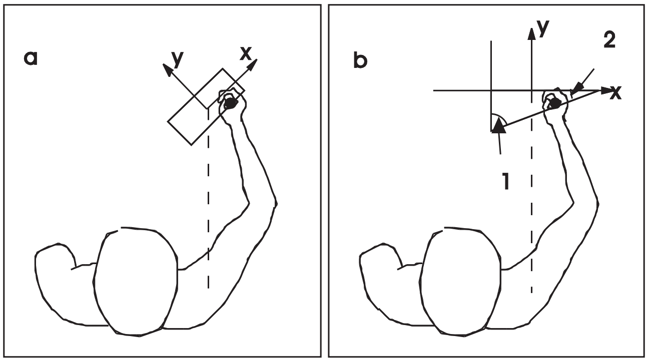

Fig. 2. a In the first experiment, subjects were asked to judge the relative lengths of simulated rectangular containers. **b** Angle perception is expected to be related to length perception. In the second experiment, subjects were asked to judge the relative magnitude of angles of simulated triangular containers

Four experiments were performed to test the hypothesis. Three experiments were performed to quantify perceptual distortion of length, angle and orientation at a single arm configuration. Subjects interacted with simple, simulated objects such as rectangular and triangular holes, as shown schematically in Fig. 2. Subjects were asked to make judgments about spatial properties of the objects: relative side lengths, relative angle magnitudes, and absolute object orientations.

A fourth experiment examined motor behavior. Given that spatial perception is distorted, it is plausible that motor behavior may be distorted in a way consistent with perceptual distortion. In the fourth experiment, subjects were asked to draw circles. The shapes drawn were compared with those predicted given the results of the length perception experiment.

1.2 Previous work: the tangential-radial distortion

The perceptual distortion most relevant to this work is the tangential-radial distortion (also referred to as the tangential-radial effect or illusion). Experimentally, it is observed that the perceived length of a line segment depends on its position and orientation with respect to the subject. In particular, line segments oriented radially from the shoulder are perceived as being longer than line segments oriented tangentially to circles centered at the shoulder (Künnapas 1955; Davidon and Cheng 1964; Day and Avery 1970; Deregowski and Ellis 1972; Day and Wong 1973; Wong 1977, 1979; von Collani 1979; Marchetti and Lederman 1983; Kay et al. 1989a,b; Fasse et al. 1990; Hogan et al. 1990; Fasse 1992). Wong (1977, 1979) hypothesized that the tangential-radial distortion is caused by a difference of inertia in the two directions; distance is inferred from temporal cues, that is, subjects try to move at a constant velocity and infer distance from time. Marchetti and Lederman (1983) claimed to disprove this hypothesis. Deregowski and Ellis (1972) showed that the perception of shape depends on the orientation of the shape. Experimentally they observed distortions in the order of 12%.

Kay et al. (1989a,b) (see also Hogan et al. 1990) conducted a series of experiments to study haptic perception using the same apparatus used in this investigation. They looked at perception of length, force and stiffness at three different locations in the workspace. Simulated, compliant objects were centered at 25%, 50% and 75% of the maximum reach from the subjects' shoulders. They observed that the amount of distortion is configuration dependent; distortion becomes more pronounced as the center of the object moves away from the shoulder.

1.3 Mathematical preliminaries

The following overview is presented to help clarify the design and interpretation of the experiments.

1.3.1 Geometrical analysis

The prior work on perceptual distortion shows that Euclidean geometry is not applicable. Some alternative mathematical tools are necessary to describe subjects' non-Euclidean percepts of simulated objects and basic Riemannian geometry is useful to this end. Riemannian geometry is a mathematically simple extension of Euclidean geometry based on a measure of alignment of tangent vectors known as an inner product. The inner product of vectors $\mathbf{v}$ and $\mathbf{w}$ is denoted $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^t G \mathbf{w}$, where G is a symmetric, positive-definite matrix. This inner product can, in theory, vary from location to location and, in general, perceptual distortion is known to be location dependent. However, in these experiments, we were concerned only with perceptual distortion in a small region and, in that case, we may assume the inner product is effectively constant. Inner products induce norms of vectors, measures of length, and measures of angle. For example, the norm of a vector $\mathbf{v}$ is the square root of the inner product of that vector with itself, $\|\mathbf{v}\| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2}$. An inner product can be used to generate geometrical shapes similar to Euclidean shapes. For example, a Riemannian circle of radius r can be identified with the set of displacement vectors from its center of length r, $\{\mathbf{v} \mid \mathbf{v}^t G \mathbf{v} = r^2\}$. This is the equation of an ellipse.

An idealized observer's perception may be characterized by the set of ellipses that are perceived as circular. Observers will be considered equivalent if they agree which ellipses are circular. (They generally will not agree about the radius of any particular ellipse.) Equivalent observers are said to belong to the same metric class. A metric class can be represented by any of the characteristic ellipses.

1.3.2 Statistical analysis

This graphic representation cannot be used to show statistical distribution; a set of coordinates is needed. Two possible coordinate systems are shown in Fig. 3.

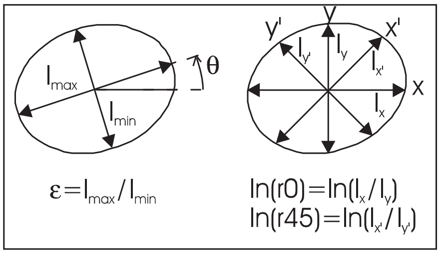

Fig. 3. Two possible sets of coordinates: on the left, ellipse eccentricity and major axis angle; on the right, ln(ratio) coordinates

An obvious choice of coordinates (shown on the left in Fig. 3) is ellipse eccentricity, ϵ, (the ratio of major and minor axis lengths) and major axis angle, θ. Unfortunately, this set of coordinates has poor statistical properties. The major axis angle is not defined for an eccentricity of one and, for eccentricities near one, the major axis angle is expected to have a nearly flat distribution. Also, the two distributions are not expected to be statistically independent; since there is more angular uncertainty for eccentricities near one, the interpretation of angular measurements is dependent on the corresponding eccentricity measurements.

A better choice of coordinates is ln(ratio) coordinates, defined on the right in Fig. 3. The ratio of lengths in any two, fixed directions is expected to have an approximately log-normal statistical distribution. Thus, the logarithm of the ratio of lengths in any two, fixed directions is expected to be approximately normally distributed. Furthermore, these distributions are expected to be statistically independent. The ln(ratio) coordinates used are denoted by $\ln(r0)$ and $\ln(r45)$.

The statistical distribution of experimental data reported here will be given in ln(ratio) plots such as that shown in Fig. 4. Each point corresponds to a metric class and can be identified with a shape such as an ellipse. The origin corresponds to the Euclidean metric class. The set of metric classes has an interesting structure, including a natural metric. This structure is discussed further by Fasse and Hogan (1993) (see also Fasse 1992).

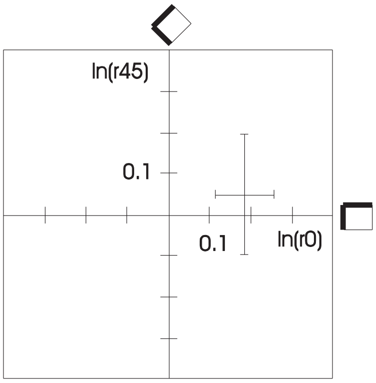

Fig. 4. ln(ratio) plots are used to show the statistical distribution. Shown is the response distribution of a single subject for the first experiment

The icons next to the $\ln(r0)$ and $\ln(r45)$ axes show which length ratios are to be compared. Variables $\ln(r0)$ and $\ln(r45)$ are assumed to be normally distributed and independent. The error bars indicate plus or minus one standard deviation from the mean.

Figure 4 also illustrates the measures of perceptual distortion and perceptual uncertainty as used in this work. Consider perception of a spatial property such as length. When asked to make judgments about spatial properties, humans will give a distribution of responses. Perceptual distortion is defined to be the systematic bias of the response distribution from the objective description. Perceptual uncertainty is defined to be a measure of the size or shape of the distribution that is independent of the systematic component. Perceptual uncertainty may be attributed to different sources. It may be the result of unquantified external influences or a temporal variation of physical properties of the perceptual apparatus. It can also be the result of inherent ambiguity of the perceptual process Bennett et al. (1989). Figure 4 depicts the response distribution of an observer exhibiting both perceptual distortion and uncertainty. This observer's perception is distorted because the means are not zero; the observer's perception is uncertain because the standard deviations are not zero.

2 Length judgment experiment

The goal of this experiment was to quantify perceptual distortion of length. This was done by simulating small rectangular containers with a robotic manipulandum and asking subjects to decide which pair of sides of the container was longer. A schematic picture of a subject interacting with a simulated container is shown in of Fig. 2a.

2.1 Method

Results are presented for eight, 20–35-year-old, right-handed subjects; seven were male, one was female. Subjects in all experiments were unpaid volunteers with backgrounds in science and engineering and with no known neurological or physical impairment.

All experiments were performed using a manipulandum designed by Fayé (1986). The manipulandum is a planar, two-link, serial linkage actuated by a pair of PMI JR16M4CH motors. The motors are driven by PMI 00-88007-003 pulse-width-modulated amplifiers. The first link is driven directly by its motor, whereas the second link is driven semi-directly (i.e., with an angular transmission ratio of 1.0) by a parallelogram linkage.

Position sensing was originally provided by Litton 70SSB 12000-1-2-1A optical encoders. These encoders are incremental encoders with phase detection circuitry capable of 8000 counts (13 bits) per revolution. The encoders were later replaced with Teledyne Gurley 25/04S-NB17-IA-PPA-QAR1S encoders, which are 17-bit absolute optical encoders. Their accuracy is comparable to their resolution.

Velocity sensing was provided by an electromechanical tachometer integral with the motor. Force sensing was provided by a Lord FT series force transducer. The manipulandum was originally controlled by a DEC PDP-11/73 computer. This was later replaced by a Dell System 325 PC.

Position and velocity were sampled continually by the controlling computer. Desired interaction forces were computed at each sampling instant based on this information and a model of the object to be simulated. These forces were generated by the actuators of the manipulandum; actual interaction forces were approximately equal to the desired interaction forces.

2.2 Procedure

Subjects were seated and grasped a vertically oriented handle on the manipulandum. Since a palmar grasp was used, subjects interacted with the manipulandum using primarily arm motion, rather than wrist or finger action. The manipulandum restricted hand motion to a horizontal plane. The height was adjusted so that the height of the subject's shoulder was approximately that of the manipulandum. Each subject's arm was supported by a sling that restricted elbow motion to a horizontal plane. A plastic panel was mounted horizontally above the manipulandum. A marker was placed on this panel to indicate the desired initial position of the hand.

The objects simulated were rectangular containers (rectangular holes in a stiff wall). In earlier experiments (Kay et al. 1989a,b; Hogan et al. 1990), the objects simulated were rectangular blocks. We found that moving along the inside of a rectangular container is easier than moving along the outside because the handle cannot slide off the corner. The rectangles were simulated as four intersecting walls. Each wall had a stiffness of 1 N/mm and a viscous damping coefficient of 10 $\mathrm{N \cdot s \cdot mm^{-1}}$ perpendicular to the wall surface. Each wall had a stiffness of zero and a damping of zero parallel to the wall surface. The stiffness and damping of the manipulandum were varied using an impedance controller (Hogan 1985). Force feedback was not used in the impedance controller so that the effective inertia of the manipulandum was the actual inertia.

The orientation of a rectangle is only defined modulo 90°; a rectangle with orientation θ is also a rectangle with orientation $(\theta + 90°)$. For this reason, only orientations of 0–90° need be considered. Rectangles were presented to the subject in two orientations. Half were presented parallel to the plane of the subject's torso (by definition 0°). The other half were presented at 45° to this plane.

Directions x and y were defined as shown in Fig. 2a. Let l_x be the length of the x side and l_y be the length of the y side. The area of the rectangles was kept at a constant $l_x l_y = 5000\,\mathrm{mm}^2$. Fifteen different aspect ratios were presented with $\ln(l_x/l_y)$ ranging from -0.40 to 0.70 in increments of $\Delta\ln(l_x/l_y) = 0.07857$.

Each container was presented for 10 s, during which time the subject traced around the bounding walls. The

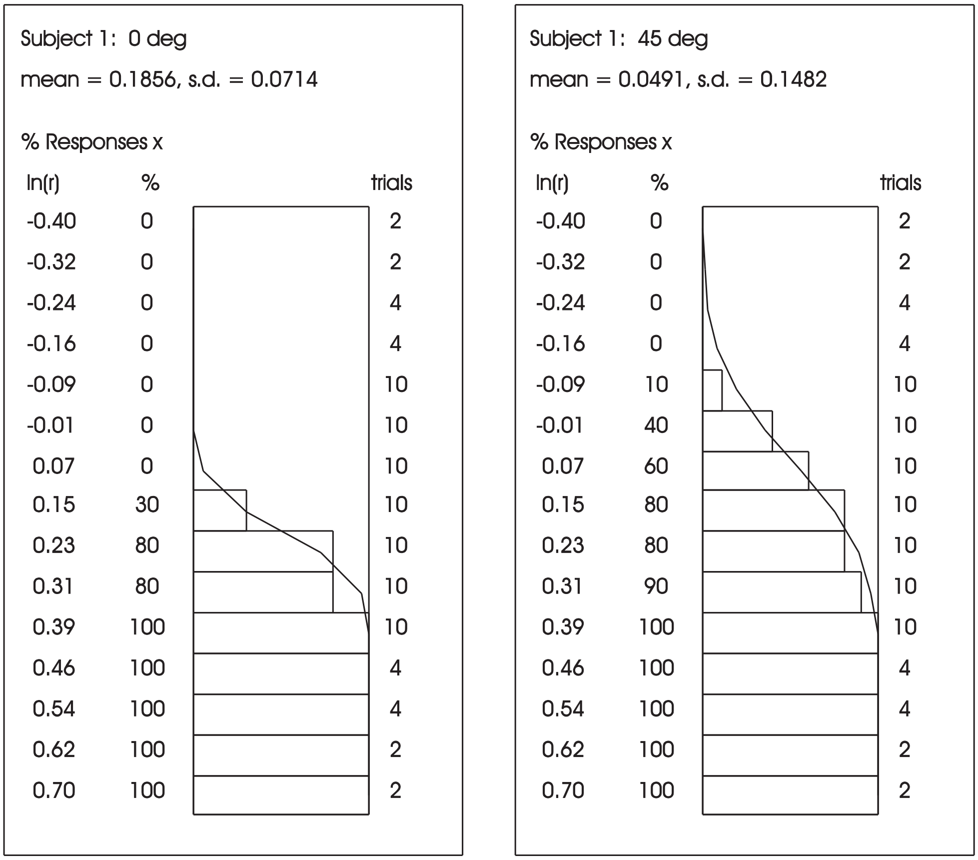

Fig. 5. Histograms of a single subject

number of times subjects interacted with each object is given in the trials column of the histograms of Fig. 5 (e.g., four 50 × 53 mm rectangles at an angle of 0°, four 50 × 53 mm rectangles at 45°, ten 45 × 57 mm rectangles at an angle of 0°, etc.). The order in which the objects were presented was random, although all subjects were presented with the same sequence of objects. In all, 188 objects were presented.

Objects in all experiments were simulated in the same region of the manipulandum's workspace. The seat position was adjusted so that simulated objects were centered at a point located in the sagittal plane on an outward radius originating at the shoulder, 75% of the distance of maximum reach from the shoulder.

Each subject was tested during a single session that lasted 1–2 h. There was a brief break halfway through the session. Subjects were instructed to keep their eyes closed during the experiment, except for brief periods between trials when they could open them to return to the starting position marked on the plastic panel above the manipulandum.

Subjects were told that they would be presented with a sequence of rectangles, that rectangles would be presented in two orientations, and that each rectangle would be available for ten seconds. Two directions named "*x*" and "*y*" were defined for each orientation. After each trial, subjects reported which side was longer, x or y.

Figure 5 shows the histograms obtained for one of the subjects. The "ln(r)" listed in the histograms is the natural logarithm of the ratio of x length to y length of the rectangle.

A cumulative Gaussian distribution function was fit to the experimental data as follows. Let i index each row of the histogram. Let p_i be the value of $\ln(r)$ in row i. Let f_i be the frequency of response observed for row i. Let n_i be the number of trials corresponding to row i. Let $\mathrm{cdf}(\bar{p}, \sigma, p)$ be a cumulative Gaussian distribution function with mean $\bar{p}$ and standard deviation σ evaluated at p. The cost function used for optimal fitting was

$$C(\bar{p}, \sigma) = \sum_i n_i [f_i - \mathrm{cdf}(\bar{p}, \sigma, p_i)]^2 \; . \tag{1}$$

The cumulative distribution function that minimizes this cost function is the function plotted in Fig. 5 along with the histogram; the mean and standard deviation associated with the fitted distribution are given above the histogram. The solution was found numerically[1].

The statistical information determined from the histograms of Fig. 5 is plotted in Fig. 4. The subject's data are represented by indicating the observed distortions [mean $\ln(r0)$, mean $\ln(r45)$] and deviations. Error bars indicate the observed standard deviations in the two directions. In this case, the subject exhibited a distortion

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[1] A more common procedure is to transform the data by mapping the interval (0,1) to the real line and to then find the minimum of a quadratic cost function of the reals. This effectively weights data points with frequencies near 0 and 1 much heavier than frequencies near 0.5. The cost function above is more appropriate.

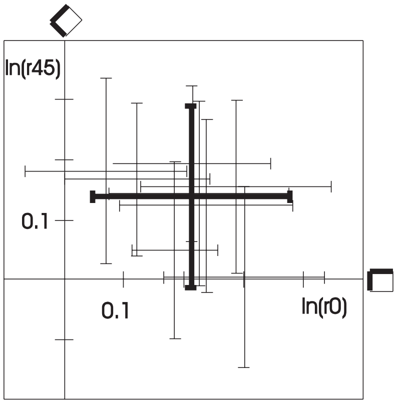

Fig. 6. Response distribution of eight subjects, together with the average distribution estimated by combining individual data

of 0.1856 at 0°, a distortion of 0.0491 at 45°, a standard deviation of 0.0714 at 0° and a standard deviation of 0.1482 at 45°.

2.3 Results

The distributions observed for all eight subjects are shown in Fig. 6. An estimate of average distortion was obtained for the eight subjects by combining all their data. The resultant distribution is highlighted in Fig. 6. The subjective circle[2] consistent with the average distortion has an eccentricity of 1.29 and a major axis angle of 17° (See Fig. 10a). The observed distortion was significant in that the mean ln(*r*0) was 1.3 standard deviations away from 0.0, and the mean ln(*r*45) was 0.9 standard deviations away from 0.0.

The results of this experiment show the following.

1. Perceptual distortion of length is significant, in the order of 30% at the location studied.
2. This perceptual distortion is similar among individuals in that all observed means of ln(*r*0) and ln(*r*45) were greater than zero.

3 Angle judgment experiment

The goal of this experiment was to test the hypothesis that a single, ideal metric relates perceived angles to perceived lengths. Subjects were presented with a number of triangles and asked to judge the relative magnitudes of two angles of the triangle. Corresponding length judgments were inferred from the subjects' responses and compared with the results of the first experiment.

To understand qualitatively how length and angle perception are expected to be related, consider the triangle of Fig. 2b, which shows schematically a subject interacting with a simulated triangle. Assume that the subject perceived the sides parallel to the x-axis and y-axis to be of equal length. This subject would be expected to perceive that angles 1 and 2 have equal magnitudes.

A plausible strategy for measuring the relative magnitude of angles of a triangle is to measure the length of the opposite sides. Allowing this direct measurement of length would have defeated the purpose of the experiment. This strategy was made impossible by preventing the subject from ever reaching the top corner. The object was simulated in such a way that the subject felt like he was tracing around a triangular hole with the curious property that the top corner could not be reached.

3.1 Method

Results are presented for seven, male, 20–35-year-old, right-handed subjects. All seven subjects participated in both the length and angle judgment experiments.

Triangles were simulated by simulating a number of intersecting walls. Each wall had a stiffness of 1 N/mm and a viscous damping coefficient of $10\,\mathrm{N \cdot s \cdot mm^{-1}}$ perpendicular to the wall and had a stiffness of zero and viscous damping coefficient of zero parallel to the wall.[3]

Subjects were prevented from measuring length directly by making it impossible to reach the third corner. This was accomplished by presenting only two sides at a time. Referring to Fig. 2b, the base side (unlabeled) was active at all times. After leaving the base, either side x or side y was active, whichever was contacted first. Returning to the base reset the state, making it possible to interact with either adjacent side. The object can be thought of as having three non-parallel sides and two corners (a biangle). There is no such Euclidean object. Since the object cannot exist physically, it must be simulated.

The length of the bases of the triangles was a constant 50 mm. Triangles were right triangles with two walls parallel to the x-axis and y-axis. Similar to the length perception experiment, triangles were presented with one wall (x) parallel to the plane going through the subjects torso or at a 45° angle to this plane. Two angles (corners), named 1 and 2, were defined for each orientation.

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[2] This is the ellipse that would be perceived to be circular by an idealized observer, one for whom length perceptual distortion for any pair of axes was consistent with the distortions measured for the pairs of axes tested in this experiment. Of course, this need not be the case for the real observer.

[3] Simulating a corner with two intersecting walls is satisfactory as long as the corner angle is right or obtuse. For acute angles, the resulting stiffness from the superimposed stiffness fields is low in the direction of the bisector of the angle, allowing significant displacements into the walls. To remedy this problem, corners were simulated by presenting three intersecting walls instead of two. The third wall was perpendicular to the bisector of the angle to be simulated. This stiffened the corner, making the corner feel more distinct. Though imperfect, the effect is much better than that achieved using only two intersecting walls.

Corner 1 was the corner opposite the side oriented in the *x* direction; corner 2 was the corner opposite the side oriented in the *y* direction.

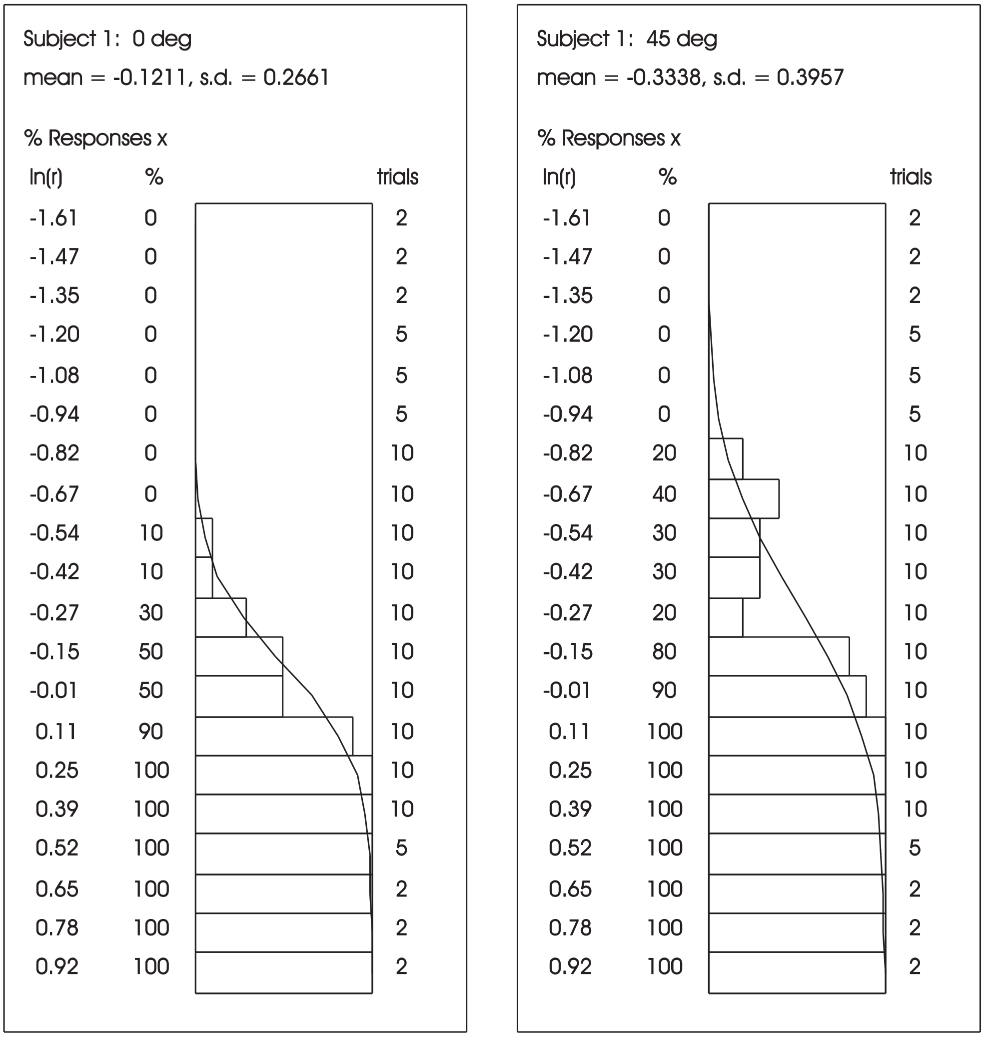
In a pilot experiment, the triangles used were those generated by bisecting the rectangles used in the length perception experiment along the diagonal. This proved to be too difficult for subjects, so a larger range of ratios was used. Let l_x be the length of the x side and l_y be the length of the y side. Nineteen different aspect ratios were presented with $\ln(l_x/l_y)$ ranging from -1.61 to 0.92 in increments of $\Delta \ln(l_x/l_y) = 0.01403$. Each object was presented for 15 s. The order in which the objects were presented was random, although all subjects were presented with the same sequence of objects. In all, 264 objects were presented.

3.2 Procedure

Subjects were tested in two sessions, with a period of 2–10 days between sessions in most cases. Each session lasted between 1–2 h. There was a brief break halfway through each session. Subjects were instructed to keep their eyes closed during the experiment, except for brief periods between trials when they could open them to return to the starting position marked on the plastic panel above the manipulandum.

Subjects were told that they would be presented with a sequence of biangles (triangles of which only two sides would be simultaneously present). They were told that the objects would be presented in two orientations, and that each object would be available for 15 s. Angles (corners) 1 and 2 were defined for each orientation. After each trial, subjects reported which angle was larger, 1 or 2.

Two histograms of responses were determined for each subject, similar to the histograms of Fig. 5. A cumulative distribution function of Gaussian type was fited to each histogram. Figure 7 shows the histograms obtained for one of the subjects. The "ln(*r*)" listed in the histograms is the natural logarithm of the ratio of x length to y length of the sides of the triangle. The "% responses x" listed in the histogram is actually the "% responses 1", but is so labeled to emphasize the expected correspondence of histograms obtained in the length and angle perception experiments.



Subject 1: 0 deg

mean = -0.1211, s.d. = 0.2661

% Responses x

ln(r)	%	trials
-1.61	0	2
-1.47	0	2
-1.35	0	2
-1.20	0	5
-1.08	0	5
-0.94	0	5
-0.82	0	10
-0.67	0	10
-0.54	10	10
-0.42	10	10
-0.27	30	10
-0.15	50	10
-0.01	50	10
0.11	90	10
0.25	100	10
0.39	100	10
0.52	100	5
0.65	100	2
0.78	100	2
0.92	100	2

Subject 1: 45 deg

mean = -0.3338, s.d. = 0.3957

% Responses x

ln(r)	%	trials
-1.61	0	2
-1.47	0	2
-1.35	0	2
-1.20	0	5
-1.08	0	5
-0.94	0	5
-0.82	20	10
-0.67	40	10
-0.54	30	10
-0.42	30	10
-0.27	20	10
-0.15	80	10
-0.01	90	10
0.11	100	10
0.25	100	10
0.39	100	10
0.52	100	5
0.65	100	2
0.78	100	2
0.92	100	2

Fig. 7. Histograms of a single subject

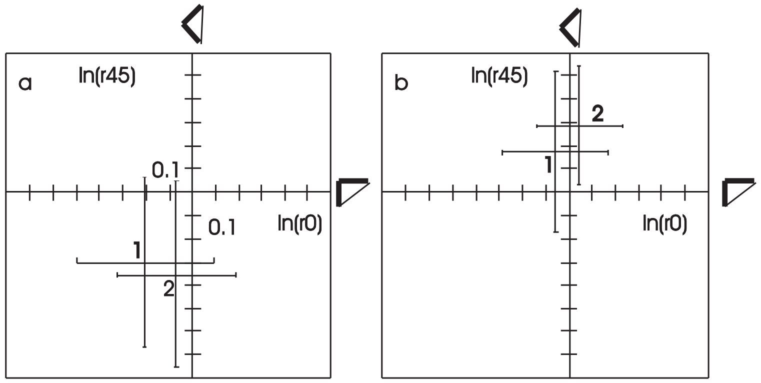


Fig. 8. **a** Two sessions 69 days apart. **b** Two sessions 8 days apart

3.3 Results

As mentioned, each experiment was split into two sets of trials. The intervals between trial sets for the seven subjects were 69, 8, 2, 10, 5, 3 and 5 days, respectively. Although there was considerable variation in the response distributions observed among subjects, the response distribution of individuals was stable over time. Subject 1's unusually long hiatus was due to an oversight; the results were still consistent. Distributions measured for two individuals on different days are shown in Fig. 8. The response distributions of all seven subjects are shown in Fig. 9. The distributions shown are those obtained by combining data from the two trial sets for each subject.

An estimate of average distortion was obtained for the seven subjects by combining all their data[4]. The average distortion was 30%. The resultant response distribution is highlighted in Fig. 9. The mean $\ln(r0)$ was 0.4 standard deviations away from 0.0, and the mean $\ln(r45)$ was 0.5 standard deviations away from 0.0. Also shown in the figure is the estimate of the average distribution obtained in the length perception experiment. The distributions are inconsistent. Both the observed distortions and uncertainties were significantly different in the two experiments [significances $P0$, $P45 < 1\%$ as determined by Student's t-test for normally distributed data sets with unequal variances on the distributions of individual mean $\ln(r0)$ and mean $\ln(r45)$ [24]]. The subjective circles consistent with the observed distortions are shown in Fig. 10a.

Figure 10b illustrates how an ellipse is related to an angle measurement. The non-Euclidean protractor on the left would be perceived as the Euclidean protractor on the right. The marks shown along the ellipse are at increments that would be perceived to be 10° apart. Since this experiment makes no predictions about the perceived orientation of a single line segment, only about the perceived angle between line segments, the marks should not be interpreted as measures of orientation.

Summarizing, the results of this experiment show the following.

1. Perceptual distortion of an angle is of the order 30% at the location studied.
2. The observed angle perceptual distortion was significantly different to that expected given the observed length perceptual distortion.
3. There is more angle perceptual uncertainty among individuals than would be predicted from the length perceptual scatter; the standard deviations were consistently higher than those of the length experiment.
4. Response distributions observed for individuals were consistent over time.

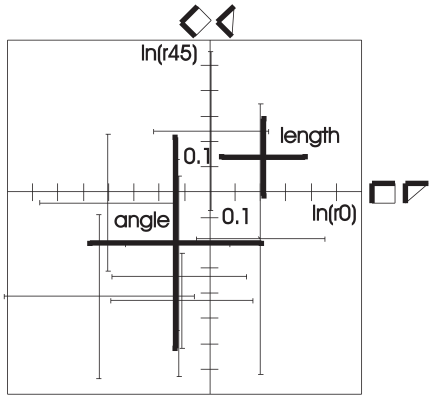


Fig. 9. Response distributions of seven subjects, together with the average distribution estimated by combining individual data. Shown also is the average distribution determined from the length experiment

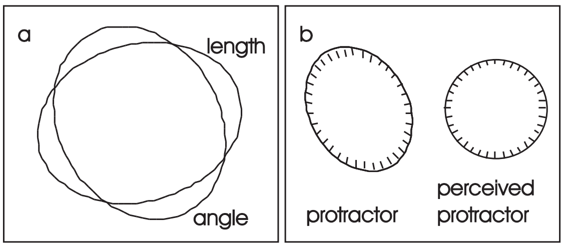


Fig. 10. **a** Average subjective circles as determined by (1) the angle experiment, with $\epsilon = 1.28$, $\theta = -62°$ and (2) the length experiment, with $\epsilon = 1.29$, $\theta = 17°$. **b** The results of the angle judgment experiment are consistent with an observer using the protractor shown on the *left*. This protractor would be perceived as the normal, Euclidean protractor shown on the *right*

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[4] A similar estimate of the means may be obtained by averaging the means across subjects, but yields no estimate of the standard deviation.

4 Orientation judgment experiment

Section 3 explained why it is reasonable to expect length and angle perceptions to be related. In experiment 3, we again looked at the consistency of length and angle perception, albeit indirectly. The goal of the experiment was to determine locally which directions are perceived as being oriented "straight ahead" and "straight sideways". This was done by simulating short slots (segments inside a solid boundary) and asking subjects to decide whether or not the end of the slot farthest away from the subject pointed forward and to the left, or forward and to the right. Figure 11a shows schematically a subject interacting with a simulated slot. The hypothesis that spatial perception is metrically consistent does not predict which directions will be perceived as "straight ahead" and "straight sideways", but it does predict that those two orientations will be perceived as being at right angles.

Knowing only one pair of orientations that are perceived to be at right angles is not enough to determine a consistent metric. For this reason, the results cannot be compared directly with those of the other experiments, and strong conclusions about metric consistency cannot be drawn. Nonetheless, indirect comparison is possible and the results are interesting.

4.1 Method

Results are presented for seven, male, 20–35-year-old, subjects. Six subjects were right-handed, one subject was left-handed. Six of the seven subjects participated in the length, angle and orientation judgment experiments.

Line segments were presented as short slots in a stiff wall. The same container simulator used in the length perception experiment was used to simulate slots. A slot was simulated as a container with zero width and finite length. Line segments were presented to subjects in 5° increments from −90° to 85°. Positive angles are measured clockwise from a sagittal orientation. Segments were all 5 cm and centered at the standard starting position. Five segments of each of the 36 orientations were presented to each subject, for a total of 180 segments. Segments were presented in a random order, but all subjects were presented with the same sequence. Each segment was available for 10 s, during which time the subject traced back and forth inside the slot.

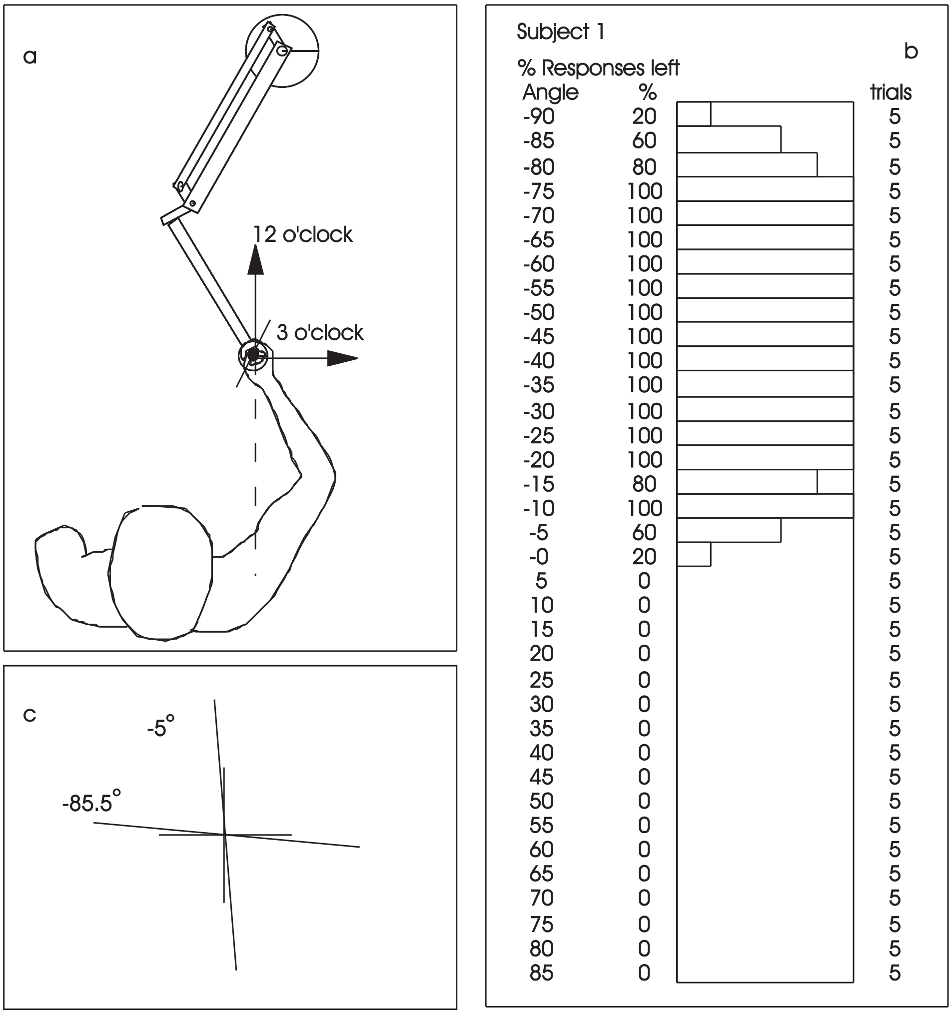

Fig. 11. a Subjects interacted with simulated slots and were asked to judge which quadrant the slot was in. **b** Response distribution of a single subject. **c** Subjective orientations determined from the response distribution of **b**

4.2 Procedure

Subjects were tested in a single session that lasted about 45 min. Subjects stopped for a brief break halfway through the experiment. Subjects were instructed to keep their eyes closed during the experiment, except for brief periods between trials when they could open them to return to the starting position marked on the plastic panel above the manipulandum.

Subjects were told that they would be presented with a sequence of slots (short line segments), each of which would be available for 10 s. They were asked to determine "whether each line segment is pointing forward and to the left, or forward and to the right of straight ahead". Subjects gave an abbreviated response of "left" or "right".

A histogram of responses was obtained for each subject. One such histogram is shown in Fig. 11b. It is probably obvious that it is difficult to judge if the slot orientation is near 12 o'clock but it may be less obvious that it is difficult to judge if the slot orientation is near 3 o'clock. There are two stimulus regions in the histogram for which the response frequency is approximately 50%; one corresponds to the subjective 12 o'clock orientation, the other to the subjective 3 o'clock orientation. These orientations were obtained using the following procedure. In each of the two crossover regions, a linear function, $f(\theta)$, was fit to the histogram using a standard linear regression. The orientation, θ_c, for which $f(\theta_c) = 50\%$, was the orientation of subjective 12 o'clock or 3 o'clock, as appropriate. Figure 11c shows the resultant orientations of subjective 12 o'clock and 3 o'clock.

4.3 Results

Figure 12a shows the subjective orientations from all seven subjects, with the data from the left-handed subject reversed. One subject exhibited a qualitatively different distribution of responses from the other subjects.

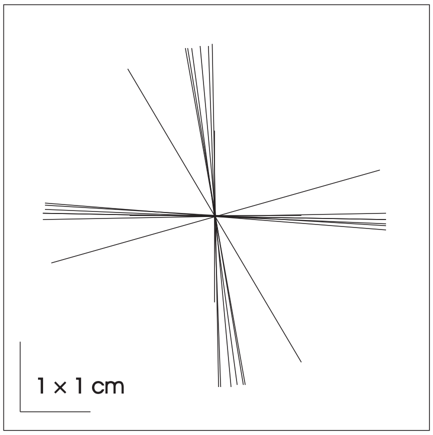

Fig. 12. Estimates of subjective 12 o'clock and 3 o'clock directions of seven subjects

We are interested in metric consistency, and not orientation perception per se. For this reason, it is only relevant that for each subject there is a known pair of segments that are (presumably) perceived to be orthogonal. Consider any two pairs of segments. For ease of visualization, assume that one pair corresponds to a "normal" subject, such as the pair shown in Fig. 11c. Assume that the other pair corresponds to the qualitatively different "outlier" of Fig. 12a. Are these responses metrically inconsistent? Not necessarily; the two subjects would disagree about the absolute orientations of each other's segments, but they might agree that both pairs of segments were orthogonal. For example, the outlier might perceive that the normal subject's segments were oriented at 2 o'colck and 5 o'clock, and thus orthogonal. It is thus not reasonable to simply average the subjects' data. Instead, to make a comparison with the other experiments, we used the results of (1) the length experiment and (2) the angle experiment to derive two predictions of the perceived angle between "straight ahead" and "straight sideways".

Predicted, perceived angles were determined as follows. A mean distortion with eccentricity of 1.29 and a major axis angle of 17° was determined in the length judgment experiment. A corresponding inner product matrix $\mathbf{G}$ is

$$\mathbf{G} = R(17^\circ)\begin{bmatrix} 1 & 0 \\ 0 & 1.29^2 \end{bmatrix} R(-17^\circ) \tag{2}$$

$$= \begin{bmatrix} 1.0568 & -0.1857 \\ -0.1857 & 1.6073 \end{bmatrix}, \tag{3}$$

where

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} . \tag{4}$$

For each subject, vectors corresponding to subjective 3 o'clock and 12 o'clock were determined. For example, the vectors corresponding to the orientations -6° and -88° are

$$\mathbf{v} = \begin{bmatrix} -0.1045 \\ 0.9945 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -0.9994 \\ 0.0349 \end{bmatrix} . \tag{5}$$

The predicted, perceived angle between these vectors is then

$$\angle(\mathbf{v}, \mathbf{w}) = \arccos\left(\frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\| \, \|\mathbf{w}\|}\right) = 75^\circ \ . \tag{6}$$

Using the inner product of (2), predicted, perceived angles were computed for each subject. These angles were averaged using circular statistical methods [1], resulting in an average of 75° (±10°, 99% confidence). This is significantly different from the expected 90°.

The results of the orientation judgment experiment are more consistent with the results of the angle perception experiment. For each subject, predicted, perceived angles were computed using an inner product matrix derived from the results of the angle perception experiment. The average of these angles was 91° (±10°,

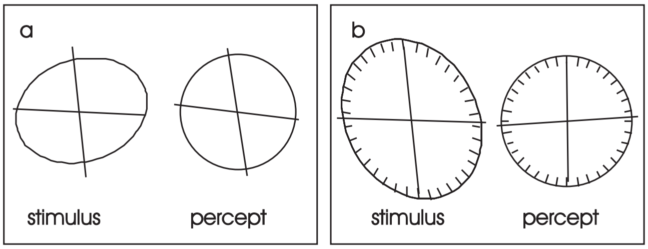

Fig. 13. **a** The percept of the stimulus is assumed to be a linear stretch of the stimulus consistent with the length perception experiment. **b** Orientation perception results are more consistent with angular perception results

99% confidence). This angle is obtuse; the acute angle is 89°. Both are statistically similar to the expected 90°.[5]

To understand these results, consider Fig. 13 which may provide qualitative insight on the implications of the statistical analysis. Figure 13a shows an idealized stimulus-percept pair. The stimulus consists of an ellipse and two line segments corresponding to the vectors of (3). An ideal observer whose perceptions were consistent with the results of the length perception experiment would perceive the ellipse to be circular. The observer would perceive the relative, acute angle between the two line segments as being 75°. Figure 13b shows a different stimulus-percept pair. The stimulus is a different ellipse with tick marks around its circumference, and again the two line segments. An ideal observer whose perceptions were consistent with the results of the angle perception experiment would perceive the tick marks to be at equal, 10° intervals. The observer would perceive the relative, obtuse angle between the two line segments as being 93°.

The conclusions drawn from this experiment are as follows.

1. The observed orientational distortion was not consistent with the observed length perceptual distortion.
2. The observed orientational distortion was consistent with the observed angular perceptual distortion.

5 Circle drawing experiment

Humans make errors when performing motor tasks such as drawing. For example, we cannot draw perfect circles without special tools. This error is often thought of as resulting from failure of the motor control system to achieve the intended behavior but alternatively, some of this error may also be ascribed to a distorted intent.

Assume, for example, that a subject was asked to draw a circle. It is reasonable to assume that it is the subject's intent to draw a shape that is perceived to be circular. If the subject's perception is distorted so that Euclidean circles are not perceived as being circular, it is unlikely that the subject will draw Euclidean circles, even if no error is introduced by the motor control system. Motor behavior may thus be expected to be related to perceptual behavior. More specifically, motor intent may be expected to be geometrically consistent with perceptual distortion. The goal of the fourth experiment was to determine if the shapes that people drew could be predicted, given the results of the length perception experiment.

5.1 Method

Results are presented for 11, male, 20–35-year-old, right-handed subjects. Two of the 11 subjects participated in both the length judgment and the circle drawing experiments.

The manipulandum was used solely as a data acquisition device in this experiment. No controller was active, so the primary influence of the manipulandum was its inertia and friction, both of which were sufficiently small to have little or no effect on movement.

5.2 Procedure

Subjects were asked to draw circles continually. The circles were to be centered at the standard starting position (75% of maximum reach), approximately 10 cm in diameter. The circles could be drawn in a clockwise or counterclockwise direction, but once a direction had been chosen, all circles were to be drawn using that direction. Nine of the 11 subjects chose to draw circles counterclockwise.

Subjects performed the experiment with their eyes closed, although they could open their eyes periodically to make sure the circles were centered properly. Data was sampled at 50 Hz and stored in a ring buffer that held 4 s of data. Movement speeds were such that at least one complete circle was drawn in 4 s. Subjects were told to slow down if their rates approached one circle per 2 s. Data collection was halted at the subject's report that a "good" circle had been drawn. Subjects drew 20 circles in all. The experiment lasted about 15 min.

Ellipses were fit to the data by finding a metric for which the data looked most circular. The procedure used is described in the appendix. Ellipses fitted using this procedure are shown in Fig. 14.

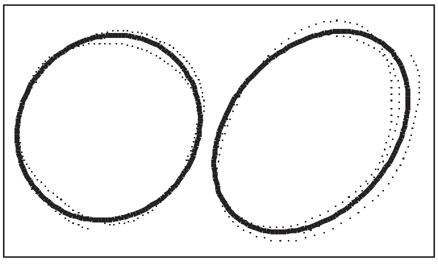

Fig. 14. Two examples of ellipses fitted to data

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[5] Note that the estimated ±10°, 99% confidence intervals of the predicted, perceived angles take into account only statistical variation of the orientation judgment data and not of the length or angle judgment data, which were assumed to be described perfectly by their means.

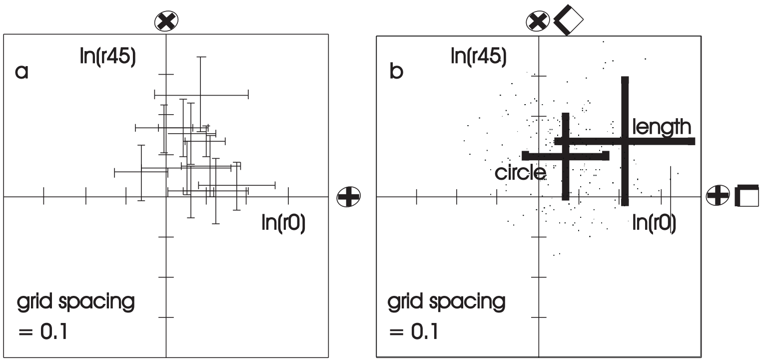

Fig. 15. **a** Ellipse distributions of 11 subjects. **b** Distributions obtained from both the circle drawing experiment and the length perception experiment. Points correspond to ellipses drawn by all 11 subjects

This procedure is sensitive to drift in the center of the data because it assumes that the center of the circles is stationary. Attempts to fit a single ellipse to data that is best described as multiple circles with different centers will lead to erroneous results. The center of the ellipses tended to drift toward the subjects' midlines. For this reason, the data recording duration was chosen so that subjects had time to draw only one or two circles.

Measures of statistical distribution analogous to those used in the perceptual experiments were obtained by assuming that the distributions of $\ln(r0)$ and $\ln(r45)$ of the ellipses were independent and distributed normally. The mean values of $\ln(r0)$ and $\ln(r45)$ thus obtained determined an average ellipse.

5.3 Results

Distributions obtained for each of the 11 subjects are shown in Fig. 15a; most of the means are in the first quadrant. Figure 15b shows the resultant distribution from combining the data from all 11 subjects. Discrete points in the plot correspond to ellipses drawn by subjects. The distribution measured in the length perception experiment is presented for comparison. Although the distributions were statistically distinguishable [significances $P0 < 1\%$, $P45 = 45\%$ as determined by Student's *t*-test for normally distributed data sets with unequal variances on the distributions of individual mean $\ln(r0)$ and mean $\ln(r45)$], the difference of means is small.

The results of this experiment show the following.

1. Motor distortion of length is significant, in the order of 13% on average.
2. This motor distortion is similar among individuals.
3. The observed motor distortion was inconsistent with the observed length perceptual distortion.

It is interesting to note that the relative distortion between the distributions obtained in the length perception and circle drawing experiments was much smaller than the relative distortion between the distributions obtained in the length perception and angle perception experiments (compare Fig. 9 with Fig. 15b). Thus, despite their inconsistency (statistical distinctness), it appears that motor distortion and length perceptual distortion may be similar.

6 Discussion

6.1 Validity of the results

The experiments reported here were performed on "virtual" objects simulated mechanically by a robot to mimic certain properties of real objects, but the mimicry is inevitably imperfect. Could our observations have been due to some difference between real and virtual objects? Further work would be required to conclusively rule out this possibility but the similarity of our length perception results to earlier investigations of the tangential-radial effect makes it unlikely.

In the length perception experiment, subjects were limited to 10 s of interaction before judgment, which might have affected the results. In the length perception experiment reported by Kay et al. (1989a,b) subjects were given unlimited time of interaction before judgment. Our results were similar to those of Kay et al. so the limited duration of interaction seems not to have been important.

Could our observations be peculiar to the experimental task we used? Task context is undoubtedly important. Haptic perception may depend upon the particular motor and sensory apparatus involved. Subjectively, this task is analogous to feeling inside a shape made of stiff foam[6] with a stick held in a palmar grasp. As humans usually explore objects with the fingers rather than with the palm, we might find different results if we confined our subjects to a fingertip pincer grasp. Once again, the similarity of our results to prior work on the tangential-radial effect (which was not restricted to palmar interaction) suggests that our observations are not peculiar to the grasp our subjects used but further work is required to investigate this possibility. Subjects were allowed to open their eyes periodically, so that they would remain alert; these intermittent visual cues may have affected the results. Furthermore, given that vision plays a dominant role in spatial perception, different results might be found if the experiments were performed with the eyes open. Without further research we cannot rule out these possibilities, but we note that there are realistic everyday situations in which haptic perception occurs in the absence of visual information. The observed haptic perceptual distortions were similar for

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[6] Due to limitations of the apparatus, objects with perfectly rigid sides could not be simulated.

different subjects and consistent over time, indicating that they reflect a robust perceptual phenomenon, interesting in its own right.

Could our observations be an artifact of our experimental design? While the order of stimulus presentation might have been varied to eliminate possible order effects, in each experiment we chose to use the same trial sequence for all subjects. This was done to enhance consistency between subjects by ensuring that all subjects experienced the same set of trials but alternative designs may be appropriate for further research.

In the orientation perception experiment, subjects were allowed only two possible responses: forward and to the left, forward and to the right. As a result, we could not determine a metric directly from this experiment and had to use the indirect method described above to test whether orientation perception was consistent with length perception (it was not) and angle perception (it was). However, if subjects were given, say, four possible responses, four subjective orientations might be determined. That would be more than enough information to determine a metric, so that it would then be possible to compare the experimental results directly. This would also make it possible to take into account the statistical variation of the length and angle judgment data, which was not possible with the indirect method we used.

Each of our experiments used a relatively small number of subjects, and not all the same subjects. For example, only two subjects participated in both the length perception and the circle drawing experiments. Finally, in the length and angle perception experiments, only two object orientations were used to determine a "perceptual metric". In effect, it was assumed, in accordance with basic Riemannian geometry, that length perception for all other object orientations could be derived from these two measurements. This, of course, need not be the case and further experimentation would be required to validate this assumption.

Nevertheless, even with these caveats, our experiments yielded statistically reliable results that admit a meaningful interpretation.

6.2 Relation to prior work

These experiments confirm and extend prior observations of a tangential-radial distortion of haptic perception. The effect is quite pronounced. For example, length perception is distorted in the order 25–30% in the workspace location studied. It becomes even more pronounced as the center of the object moves away from the shoulder (Kay et al. 1989a,b; Hogan et al. 1990). However, it appears that the effect is not best represented as a combination of radial and tangential effects. Our results indicate that the greatest length perception distortion (indicated by the minor axis of the ellipse of Fig. 10) is not radial (i.e., along a radius centered on the shoulder) but in a direction oriented more towards midline. Contrary to the hypothesis proposed by Deregowski and Ellis (1972) we observed significantly distorted perception of objects at 45°.

Our experiments also extend prior work to quantify haptic perception of other spatial properties of objects such as angles of corners. We found that haptic angle perception was also significantly distorted, in the order of 30% in the workspace location studied.

By quantitatively comparing the perceptual distortion of different object properties such as angles and lengths, we were able to investigate the geometric structure of haptic perception. We believe this fundamental approach may ultimately reveal important features of the computations underlying perception.

6.3 Geometric inconsistency

The first result of our study is experimental evidence that haptic spatial perception was not geometrically consistent or, more specifically, not consistent with a Riemannian mathematical model. However, this does not imply that haptic perception is unstructured. The second result of our study is our experimental evidence that perception of orientation was largely consistent with perception of angle. This suggests that there may be more commonality between the computational resources (or modules, or regions of the CNS) used in perceiving orientation and relative angle than between those used in perceiving orientation and length. There appears to be an internal orientation perceptual apparatus unrelated to the apparatus used to perceive length and distance. Put another way, there appears to be an internal, metrically inconsistent compass.

What does it mean for a compass to be metrically inconsistent? A compass is a device for measuring the absolute orientation of a single line segment. A protractor is a metrically consistent device for measuring the relative angle between pairs of line segments. The compasses used in everyday navigation are metrically consistent in that there exist metrically consistent angular relations between the various bearings on the compass. For example, the orientation East is at 90° to the orientation North, which is in turn at 45° to the orientation Northwest. Metrical consistency is not a logical necessity, it is a logical convention. It is possible to navigate using compasses that are not metrically consistent.[7]

To our knowledge, there is no obvious reason why humans would have developed an apparatus for measuring angles between line segments (an internal protractor). There is, however, ample reason to expect that humans would have developed an orientation perceiving apparatus (an internal compass) as it would be useful in making directed movements.

6.4 Perception of length and production of movement

Since haptic perception requires an interplay between sensation and action, we studied the relation between

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[7] The sidereal compass used for navigation in certain Micronesian cultures is an example of a metrically inconsistent compass (Hutchins 1983).

haptic perceptual distortions and movement production. The third result of our study is experimental evidence that distortion of motor production is similar to the distortion of length perception (though a small, statistically significant difference was observed). This is further evidence of geometric structure in haptic perception. It also implies that commonly observed errors in production of motor behavior may, at least in part, be attributed to central processes underlying the production of motor commands rather than to an imperfect execution of motor commands.

One interpretation consistent with all of our experimental results is that there may exist both an "internal compass" and an "internal ruler". The internal ruler is used in length perception; the internal compass is used in orientation perception. The compass and the ruler are functionally independent, or the results of the length and orientation perception experiments would have been consistent. In estimating relative angles between segments, humans refer to their internal compass, and not their internal ruler. This is supported by the geometric consistency between the angle and orientation perception experiments. In the generation of paths, humans refer to their internal ruler. This is supported by the similar geometric distortion observed in the length perception and circle drawing experiments.

This hypothesis is in accord with results in motor neuroscience that show that direction of movement is encoded independent of velocity, notably the work of Georgopoulos and colleagues (Kalaska et al. 1983; Georgopoulos et al. 1986, 1988; Taira et al. 1990; Georgopoulos 1991). It is also consistent with the phenomenon of dysmetria, where individuals move the hand in an appropriate direction towards a target, but overshoot or undershoot the distance (Meador et al. 1986). Of course, the results of the experiments presented here are not strong enough to verify these speculations but further investigation is clearly warranted to explore the ramifications of these ideas and test the hypothesis that independent CNS processes are used for orientation and length perception.

7 Concluding remarks

In the experiments reported here, we found that humans misperceive geometric properties of felt objects. We believe it is important to understand and quantify this phenomenon in order to better design and develop devices to interact with humans. This is especially relevant for devices to simulate the touch and feel of real objects, an important part of virtual environment technology. Furthermore, the analytical and experimental methods described here may prove useful for objective assessment of virtual environment technology, e.g., the "real-ness" of a particular display (see Fasse and Hogan 1993, for further discussion).

Conversely, our experiments illustrate the value of virtual environment technology for psychophysical investigations. Even the relatively crude device we used afforded a useful enhancement of psychophysical methods previously used to investigate haptic perception. By providing enhanced control of experimental stimuli, virtual environment technology provides significant new opportunities for studying the transformation from sensation to action.

Finally, the psychophysical techniques used in these experiments are potentially useful for the evaluation of patients with neurological disorders, for example, recovering stroke patients. A "haptic examination", analogous to a vision or hearing examination, could be administered using simple robot technology (see, e.g., Krebs et al. 1998) and may provide useful insight into the nature and extent of the perceptual abilities and disabilities of patients. Ultimately, a haptic examination might be useful for clinical diagnosis or evaluation of recovery.

Acknowledgements. The research reported in this paper was supported in part by National Institutes of Health Grant AR40029, Office of Naval Research Grant N00014/88/K/0372, and by the Fairchild Foundation. E.D. Fasse is now with the Dept. of Aerospace and Mechanical Engineering at the University of Arizona. B.A. Kay is now with the Dept. of Cognitive and Linguistic Sciences at Brown University. F.A. Mussa-Ivaldi is now with the Dept. of Physiology at the Northwestern University Medical School.

Appendix: Ellipse fitting

This appendix describes the procedure that was used to fit ellipses to experimental data. Ellipses were fit to data by finding a metric for which the data was most circular. Let G be a matrix representing an arbitrary metric, so that

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^t G w = \mathbf{v}^t \begin{bmatrix} g_{xx} & g_{xy} \\ g_{xy} & g_{yy} \end{bmatrix} \mathbf{w} \ . \tag{7}$$

A circle with respect to this metric, centered at c with radius r, is the set of points $\{p \mid (p-c)^t G(p-c) = r^2\}$. Real data will not be circular with respect to any metric. The deviation from being circular for a data point p_i can be expressed by

$$\begin{aligned} \epsilon_i &= (\|p_i - c\|^2 - r^2)^2 \qquad (8) \\ &= (g_{xx}p_{xi}^2 - 2g_{xx}c_x p_{xi} + g_{xx}c_x^2 + 2g_{xy}p_{xi}p_{yi} \\ &\quad - 2g_{xy}c_x p_{yi} - 2g_{xy}p_{xi}c_y + 2g_{xy}c_x c_y \\ &\quad + g_{yy}p_{yi}^2 - 2g_{yy}c_y p_{yi} + g_{yy}y_c^2 - r^2)^2 \ . \qquad (9) \end{aligned}$$

The cost function used for optimal fitting was $V = \sum_i \epsilon_i$. Define $a_0 = g_{xx}$, $a_1 = 2g_{xy}$, $a_2 = g_{yy}$, $a_3 = -2g_{xx}c_x - 2g_{xy}c_y$, and $a_4 = -2g_{xy}c_x - 2g_{yy}c_y$. Define $a_5 = g_{xx}c_x^2 + 2g_{xy}c_x c_y + g_{yy}c_y^2 - r^2 = \|c\|^2 - r^2$. Substitution into (9) yields

$$V = \sum_i (a_0 p_{xi}^2 + a_1 p_{xi}p_{yi} + a_2 p_{yi}^2 + a_3 p_{xi} + a_4 p_{yi} + a_5)^2 \ . \tag{10}$$

As stated, the problem is overspecified. There are six parameters, a_0 through a_5, which are, in turn, functions of g_{xx}, g_{xy}, g_{yy}, c_x, c_y and r^2. An ellipse requires only five parameters for description. The radius, r, can be chosen arbitrarily without affecting the solution. Unfortunately, the resultant optimization problem does not have an obvious closed-form solution and has to be solved iteratively. This approach was tried initially using a gradient descent algorithm, but was found to give unsatisfactory results on elliptical test patterns. The convergence properties of the algorithm were poor, and at times unstable. The algorithm could not be tuned to give satisfactory estimates of the elliptical test patterns.

If instead $a_5 = \|c\|^2 - r^2$ is chosen arbitrarily, the problem has a closed-form solution, as given below. Empirically, this procedure gave better estimates in general, and found the exact solutions of elliptical test patterns in particular. This is the procedure that was ultimately used for fitting ellipses to experimental data. Although the resultant fits were empirically excellent, this procedure should be used with caution. It is not obvious how fixing $\|c\|^2 - r^2$ affects the solution. Assume, then, that a_5 is constant. Let

$$a = [a_0 \;\; a_1 \;\; a_2 \;\; a_3 \;\; a_4]^t \ , \tag{11}$$

$$b = -\sum_i [p_{xi}^2 \quad p_{xi}p_{yi} \quad p_{yi}^2 \quad p_{xi} \quad p_{yi}]^t \tag{12}$$

and

$$M = \sum_i \begin{bmatrix} p_{xi}^4 & p_{xi}^3 p_{yi} & p_{xi}^2 p_{yi}^2 & p_{xi}^3 & p_{xi}^2 p_{yi} \\ p_{xi}^3 p_{yi} & p_{xi}^2 p_{yi}^2 & p_{xi} p_{yi}^3 & p_{xi}^2 p_{yi} & p_{xi} p_{yi}^2 \\ p_{xi}^2 p_{yi}^2 & p_{xi} p_{yi}^3 & p_{yi}^4 & p_{xi} p_{yi}^2 & p_{yi}^3 \\ p_{xi}^3 & p_{xi}^2 p_{yi} & p_{xi} p_{yi}^2 & p_{xi}^2 & p_{xi} p_{yi} \\ p_{xi}^2 p_{yi} & p_{xi} p_{yi}^2 & p_{yi}^3 & p_{xi} p_{yi} & p_{yi}^2 \end{bmatrix} \ . \tag{13}$$

The optimal a is $a = M^{-1}b$. Knowing a it is easy to solve for G and c, which fully describe the ellipse. Ellipses fitted using this procedure are shown in Fig. 14.

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