

Discriminating rigid from nonrigid motion: Minimum points and views

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Theoretical investigations of structure from motion have demonstrated that an ideal observer can discriminate rigid from nonrigid motion from two views of as few as four points. We report three experiments that demonstrate similar abilities in human observers: In one experiment, 4 of 6 subjects made this discrimination from two views of four points; the remaining subjects required five points. Accuracy in discriminating rigid from nonrigid motion depended on the amount of nonrigidity (variance of the interpoint distances over views) in the nonrigid structure. The ability to detect a rigid group dropped sharply as noise points (points not part of the rigid group) were added to the display. We conclude that human observers do extremely well in discriminating between nonrigid and fully rigid motion, but that they do quite poorly at segregating points in a display on the basis of rigidity.

Human observers report seeing three-dimensional (3-D) relationships in certain changing two-dimensional (2-D) images—as, for example, in images that represent projections of rotating solid objects (Wallach & O'Connell, 1953) or projections of rotating patterns of dots (Braunstein, 1962; B. F. Green, 1961). There has been recent interest in the minimum numbers of points and views that must be presented in order for subjects to make accurate judgments about 3-D structure from 2-D images. This interest stems in part from theoretical analyses of the minimum conditions under which an ideal observer can infer 3-D structure from 2-D coordinates. In this paper, we relate psychophysical data to theoretical analyses for a particular judgment: the discrimination of rigid from nonrigid motion.¹

Lappin, Doner, and Kottas (1980) studied the ability of subjects to judge 3-D relationships on the basis of only two views. They added noise to polar projections of rotating rigid spheres by varying the number of points that were in correspondence between the views. They concluded that two views were sufficient for discriminating between different levels of noise applied to rigid structures. Braunstein, Hoffman, Shapiro, Andersen, and Bennett (1987) asked subjects to discriminate between same and different rigid structures on the basis of from two to six views of from two to five points. They found that human performance exceeded theoretical expectations, al-

though some of the accuracy may have resulted from subjects exploiting the correlation that exists between 3-D and 2-D interpoint distances: 2-D interpoint distances tend to be more similar for two projections of the same 3-D object than for two projections based on different 3-D objects.

Todd (1988) has provided further evidence that two views are sufficient for distinguishing between rigid and nonrigid motion. He had subjects rate the rigidity of the depicted motion for two, four, or eight views of 14 connected line segments. The nonrigid displays were created by having each line segment's end point rotate about an axis whose position and orientation with respect to the picture plane was selected at random. The mean ratings given by subjects for nonrigid and rigid displays were at opposite ends of a 5-point rating scale. This clear discrimination between rigid and nonrigid displays did not increase with views, possibly because the effect had already reached a ceiling in the two-view condition.

In psychophysical experiments based on Ullman's (1979) theorem (that 3-D structure can be recovered from three views of four noncoplanar points), Petersik (1987) studied discrimination of rigid from nonrigid motion and found that subjects could indeed make that discrimination with three views of four points. This study included only rotations about a vertical axis. Nonrigid motion was produced by taking rigid displays and displacing points horizontally or vertically in the 2-D projection. This method, however, does not provide a clear indication of a subject's ability to discriminate rigid from nonrigid motion. When nonrigid displays are produced by perturbing the 2-D trajectories of points in a rigid display, it may be possible to distinguish between rigid and nonrigid displays on the basis of the trajectories of individual points. The most obvious case is that of a parallel projection of dots rotating about a vertical axis with a perturbation inserted in the vertical direction. All of the unperturbed

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The stimulus onset asynchrony (SOA) between views was 400 msec. There was no interstimulus interval between views. In order to allow sufficient time for subjects to make a judgment, the two views were repeated until the subject responded, up to a maximum of 60 sec.

Apparatus. The stimuli were presented on a Hewlett-Packard Model 1321B X-Y Display with a P-31 phosphor, under the control of a PDP-11/83 computer. The maximum projected diameter of each simulated object occupied 821 plotting positions on the screen and subtended a visual angle of 2°. Points were refreshed at a rate of 17.5 Hz. The dot and background luminances at the screen were approximately 5 and 0.02 cd/m², respectively. Subjects viewed the displays through a tube that limited the field of view to a circular area 7.9° in diameter. A 0.5 neutral-density filter was inserted in the tube to remove any apparent traces on the CRT. The eye-to-screen distance was 1.7 m.

A metal and plastic model consisting of four white spheres rigidly connected by thin black rods was used to instruct the subjects. The subjects responded by pressing one of two switches, one labeled "rigid" and the other "nonrigid." The responses (and response latencies) were recorded by the PDP-11/83.

Procedure. Each subject participated in one practice session followed by four experimental sessions. Each session began with 9 practice trials followed by a random sequence of 120 trials, consisting of 20 signal and 20 noise trials at each of the three point levels. The trials were presented in three blocks of 43 trials each. There was a 2-sec delay between each trial and a 1-min rest period between each block.

Subjects were instructed to press the "rigid" switch if the display consisted of a group of dots that was moving rigidly and to press the "nonrigid" switch otherwise. A group of dots was defined as moving rigidly if "the distance from any dot to any other dot remains the same, no matter how the group is moved." The model was used to demonstrate the rigid group condition. Subjects who were to receive feedback were told that a single tone would indicate a correct response and that two tones would indicate an incorrect response. The room was darkened 2 min before the trials began.

Results

A signal detection paradigm (D. M. Green & Swets, 1966) was used to analyze the results, with the trials containing a rigid group serving as signal trials. (We consider some of the implications of this definition of signal trials in the Discussion section.) A d' measure was computed for each subject and stimulus condition, using the proportion of rigid group responses on signal (3-D rigid display) trials as the hit rate and the proportion of rigid

group responses on noise (no rigid group) trials as the false-alarm rate. Each d' was based on 160 trials, half of which were signal trials.

The significance of the d' scores was calculated for each subject and number of points, using Marascuilo's (1970, pp. 238-240) one-signal significance test. Table 1 lists these d' values. Of a total of 18 d' 's (6 subjects, three numbers of points) 14 were significantly different from zero ($p < .05$). For feedback subjects, 8 (of a total of 9) were significant. For nonfeedback subjects, 6 (of a total of 9) were significant. The d' 's for all feedback subjects and for one nonfeedback subject were significant at two views of four points. The d' 's for all subjects were significant at two views of five points. The mean d' for the subjects given feedback was higher than for those not given feedback (0.84 vs. 0.67) and lower for four points (0.51) than for five and six points (0.90 and 0.85), but these differences were not statistically significant.

A measure of 3-D nonrigidity was developed to determine whether the amount of 3-D nonrigidity in the noise displays affected the d' results. This measure was the mean across pairs of points of the variances of the 3-D interpoint distances across views. (Specifically, let $\mathbf{p}_{ij} = (x_{ij}, y_{ij}, z_{ij})$ denote the position in space of point i in view j . Let $d_{i'j}$ be the 3-D distance between \mathbf{p}_{ij} and $\mathbf{p}_{i'j}$. Let $\sigma^2_{i'j}$ be the variance of $d_{i'j}$ over all views j . Then our 3-D nonrigidity measure is the mean of the $\sigma^2_{i'j}$ for all distinct i and i' .) The nonrigid displays were separated into two categories—*high* and *low* 3-D nonrigidity—according to whether nonrigidity was greater than or less than the median value. The proportion of false alarms was calculated separately for each category. The proportion of correct responses for the entire rigid group was used to calculate the hit rate. This provided separate measures of d' for nonrigid displays with low and high amounts of nonrigidity. Fifteen (of 18) d' 's were significantly different from zero when the high nonrigidity displays were used in calculating the false-alarm rate, and 8 (of 18) were significantly different from zero when the low nonrigidity displays were used. The d' values were higher for the high nonrigidity displays than for the low nonrigidity displays in 16 of 18 comparisons (6 subjects \times 3 numbers of points). The mean d' 's for the high nonrigidity and low nonrigidity displays were 0.99 and 0.54, respectively.

These results indicate that human observers can discriminate rigid from nonrigid structures at or near the minimum level at which this discrimination is theoretically possible: two views of four points. (This is the minimum level if one assumes orthographic projection and if no constraints other than rigidity are applied.) The discriminability of rigid from nonrigid motion depends on the nonrigidity in the noise trials, as reflected in our 3-D nonrigidity measure.

EXPERIMENT 2

In Experiment 2, we examine accuracy in the four-point condition as the number of views increases. Previous

Table 1
 d' Scores in Experiment 1

Subject	Number of Points		
	4	5	6
Feedback Group			
F.	0.865*	1.235*	1.635*
A.	0.550*	0.735*	0.280
T.	0.505*	0.800*	0.925*
No Feedback Group			
G.	0.345	0.715*	1.060*
L.	0.475*	1.210*	0.805*
O.	0.290	0.705*	0.405

* $p < .05$.

udies present mixed results for the effects of number of views on judgments related to recovery of 3-D structure and discrimination of rigid from nonrigid motion. Joner, Lappin, and Perfetto (1984) found increased accuracy with increasing numbers of views in discriminations between different levels of spatiotemporal correlation in polar projections of rotating dot spheres. Braunstein et al. (1987) found increasing accuracy with increasing numbers of views in discriminations between same and different 3-D structures. On the other hand, Todd (1988) found no increase in the discriminability of rigid from nonrigid structures as the number of views was increased beyond two. Theoretically, two views do contain sufficient information for discriminating rigid from nonrigid structures (Bennett, Hoffman, Nicola, & Prakash, 1989; Ullman, 1977), but a third view is required before a specific rigid structure can be recovered (Ullman, 1979). It is possible that human observers are more accurate in discriminating rigid from nonrigid motion when there is sufficient information to recover a specific structure. If this is the case, an increase in accuracy should be expected in the three-view over the two-view condition.

Number of views, however, cannot be studied in isolation. Only two of the following three variables can be held constant as the number of views is varied: (1) rate of presentation of the views, (2) amount of rotation between views, and (3) total amount of rotation in the sequence of views. We chose to hold the first two variables constant and to allow the total amount of rotation to vary with number of views. For our nonrigid displays, this resulted in an increase in our measure of 3-D nonrigidity with increasing numbers of views. It is thus possible that an increase in d' with increasing views could be attributed to an increase in nonrigidity in the noise trials (suggested by Todd, personal communication, May 1, 1989). If the effect of number of views was due to the increase in 3-D nonrigidity in the noise trials, we would expect that d' would increase steadily with increasing numbers of views, and that the increase in d' would result from a decrease in the false-alarm rate rather than an increase in the hit rate.

Method

Subjects. The subjects were 4 of the 6 subjects who had served in Experiment 1. Two subjects had received feedback in Experiment 1 and 2 had not.

Design. We examined two independent variables: number of views (2, 3, 4, 5, or 6) and SOA (66 or 400 msec). (Two levels were used, because Todd, Akerstrom, Reichel, and Hayes, 1988, found an interaction between number of views and SOA in determining ratings of rigidity.) All displays contained four points. Each subject responded to 60 signal trials and 60 noise trials at each of the 10 combinations of SOA and number of views.

Stimuli. The method of generating the stimuli was the same as that used in Experiment 1, with the following exceptions: The SOAs were 66 msec and 400 msec. The refresh rate for both SOAs was 15 Hz. The angles of rotation between views were randomly selected from a uniform distribution over integer values between 5° and 9°. For rigid displays having more than two views, a new axis of rotation was randomly selected for each additional view. This was done

so that the presence of different axes of rotation in a sequence could not be used to distinguish rigid from nonrigid displays. For nonrigid displays having more than two views, a new axis of rotation was selected for each point in each additional view.

An ANOVA was conducted on the stimulus displays, using the 2-D nonrigidity measure as the dependent variable. The independent variables were 3-D rigidity, SOA, and number of views. The 2-D nonrigidity was significantly different for the 3-D rigid and 3-D nonrigid displays [$F(1,59) = 10.8, p < .01$]. The 2-D nonrigidity measure increased significantly with number of views [$F(4,236) = 178.3, p < .01$]. There were no other significant effects or interactions. The significant effect of 3-D nonrigidity indicates that it was theoretically possible for subjects to discriminate 3-D rigid from 3-D nonrigid displays on the basis of 2-D nonrigidity. This seems unlikely, however, since the variance in the 2-D nonrigidity measure accounted for by 3-D nonrigidity was 0.3%, compared with 38.2% accounted for by number of views. The means of the 2-D nonrigidity measures were .0053 for the 3-D rigid displays and .0058 for the 3-D nonrigid displays. The means for the displays with two to six views were .0012, .0029, .0053, .0077, and .0107, respectively. The units are squared distances in a unit sphere.

Procedure. Each subject participated in 1 practice session followed by 10 experimental sessions. Each session began with 9 practice trials followed by a random sequence of 120 trials, consisting of 12 signal and 12 noise trials at each of the 5 view levels. The trials were presented in three blocks of 43 trials each. Half the experimental sessions were at the short SOA, the other half at the long SOA. The order of SOAs was alternated between sessions, with half the subjects beginning with the long SOA and the other half beginning with the short SOA. The procedure was otherwise the same as in Experiment 1.

Results

A d' was computed for each subject and stimulus condition (Table 2). For the short SOA, 15 of the 20 d' 's were significantly different from zero ($p < .05$). Of the five that were not significant, three were at the two-view level and two were at the three-view level. For the long SOA, all 20 d' 's were significantly different from zero ($p < .05$).

A two-way ANOVA was conducted with SOA and number of views as the independent variables. There were two significant effects. The main effect of SOA [$F(1,3) = 16.83, p < .05, \omega^2 = 0.08$] showed an increase in d' with longer SOA (1.46 vs. 1.09). The main effect of number of views [$F(4,12) = 29.16, p < .01, \omega^2 = 0.44$] showed

Table 2
 d' Scores in Experiment 2

Subject	Number of Views				
	2	3	4	5	6
66-msec SOA					
F.	0.905*	1.075*	1.530*	1.620*	2.030*
A.	0.200	0.390	0.460*	1.190*	1.315*
G.	0.000	0.260	1.315*	1.470*	1.575*
L.	0.490	1.630*	1.210*	1.810*	1.295*
400-msec SOA					
F.	1.190*	1.520*	2.225*	2.300*	3.035*
A.	0.670*	0.860*	1.110*	1.165*	1.745*
G.	0.715*	1.120*	1.400*	1.045*	1.460*
L.	0.825*	1.330*	1.620*	1.420*	2.495*

Note—SOA = stimulus onset asynchrony. * $p < .05$.

an increase in d' with greater numbers of views. Post hoc comparisons (Tukey's *HSD* test) showed significant differences for two views versus three, four, five, and six views; three views versus five and six views; and four views versus six views.

As in Experiment 1, d' s were calculated with the nonrigid displays divided into high and low 3-D nonrigidity subgroups. For the high nonrigidity displays, 36 of 40 d' s were significantly different from zero, with a mean d' of 1.50. For the low nonrigidity displays, 29 of 40 were significantly different from zero, with a mean d' of 1.07. The d' values were greater for the high nonrigidity displays than for the low nonrigidity displays in 37 of 40 comparisons (4 subjects \times 2 SOAs \times 5 numbers of views).

The relationship between number of views and 3-D nonrigidity, d' , hit rate, and false-alarm rate is shown in Figure 1. The 3-D nonrigidity measure increased with number of views. There was a corresponding decrease in the false-alarm rate. The hit rate remained constant, indicating that the increase in d' was due to a decrease in the false-alarm rate. This is the pattern of results that would be expected if the effect of number of views was due to the increase in the 3-D nonrigidity that occurred with increasing numbers of views. This provides a further indication of the subjects' sensitivity to variations in 3-D nonrigidity and confirms the usefulness of the 3-D nonrigidity measure as a predictor of performance in discriminating rigid from nonrigid motion.

EXPERIMENT 3

Two orthographic views of four points are theoretically sufficient to determine whether or not a 3-D motion is rigid (Bennett, Hoffman, Nicola, & Prakash, 1989; Ullman, 1977), and the results of Experiments 1 and 2 indicate that subjects can make this discrimination at these minimum levels of points and views. For displays containing more than four points, the same theoretical analysis can be used to determine whether a display contains any subset of four points that is moving rigidly. It is important to know whether subjects can also determine whether rigid motion is present under these conditions; the usefulness of a rigidity constraint would be severely limited if such a constraint could be applied only when all moving elements were part of the same rigid structure. Experiment 3 included displays in which four points were moving rigidly but which, in addition, had from one to four points that were not part of the rigid structure. The subject's task, rather than indicate whether the observed structure was rigid or nonrigid as in Experiments 1 and 2, was to determine whether the display contained at least four points that moved together rigidly.

Method

Subjects. The subjects were 3 of the 4 subjects from Experiment 2 and 1 graduate student who had not served in Experiments 1 or 2. Three of the subjects were naive as to the purposes of the experiment; 1 subject was the third author. As a precondition for par-

ticipating in this experiment, each subject was required to achieve a d' of 1.2 or better in a screening session, in which they responded to 100 trials of 12 views of four points. This criterion assured that subjects were performing, on trials with no noise points, at a level comparable to performance in Experiment 2. One of the 4 subjects failed to meet the criterion in the first screening session but succeeded in doing so in a second screening session.

Design. We examined two independent variables: number of views (2, 3, 4, or 12) and number of noise points (0, 1, 2, 3, or 4). Each subject responded to 60 signal trials and 60 noise trials at each of the 20 combinations of number of views and noise points.

Stimuli. The method of generating the stimuli was the same as that used in Experiments 1 and 2, with the following exceptions: The 2-D minimum motion criteria for a display had to be met for each point for at least one transition between views rather than for all transitions. This change was made because of difficulty in generating 12-view displays that satisfied the more stringent criterion. Also, there was a change of two parameters: SOA and range of rotation angle for transitions. Two SOAs were used, 80 msec and 240 msec. (These were selected on the basis of Todd's observations, personal communication, November, 1988, and our own observations of the SOAs required for perception of smooth motion for two-view and multiple-view displays.) The refresh rate for both SOAs was 12.5 Hz. The long SOAs were used for the two-view displays and the short SOA for the 3-, 4-, and 12-view displays. The angles of rotation were randomly selected from a uniform dis-

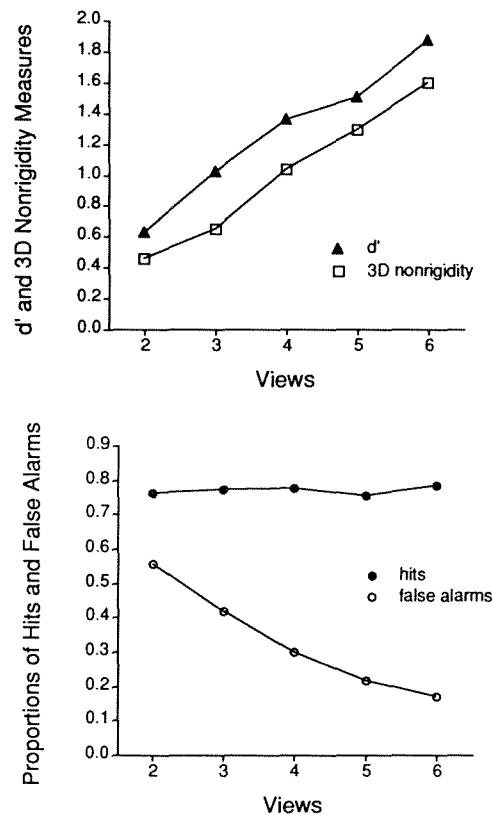


Figure 1. d' , 3-D nonrigidity, proportion of hits, and proportion of false alarms as functions of the number of views (Experiment 2). (In order to use the same ordinate values for d' and the 3-D nonrigidity measure, the nonrigidity measure is multiplied by 1,000 in this figure and in Figures 2 and 3.)

tribution of integer values between 5° and 7°. The larger rotation angles used in the previous experiments were eliminated, because they appeared to interfere with the perception of smooth motion at the 80-msec SOA.

The method used to generate the noise points added to the rigid displays was the same as that used to generate the points in the non-rigid displays, with the following additional restrictions: For each pair of views, the angle of rotation of the noise points was the same as that of the rigidly moving points, but no noise point was rotated about the same axis as that of the rigidly rotating points.

An ANOVA was conducted on the 2-D nonrigidity measure, with 3-D rigidity, number of views, and number of noise points as the independent variables. The only significant effect was the main effect of number of views [$F(3,177) = 1,510.0, p < .01$]. The means for 2, 3, 4, and 12 views were 0.0009, 0.0019, 0.0029, and 0.0133.

Procedure. Each subject participated in one or more screening sessions (described above), one practice session, and 24 experimental sessions. Each experimental session began with 5 practice trials followed by a random sequence of 100 trials, consisting of 10 signal and 10 noise trials at each of the 5 noise point levels. The trials were presented in three blocks of 35 trials each. There were 6 sessions at each of the 4 levels of number of views. The number of views across the 24 sessions was in the order 12, 4, 3, 2, 2, 3, 4, and 12 views, repeated three times.

As in Experiment 1, there was a 2-sec delay between each trial and a 1-min rest period between each block. The subjects were instructed to press the "rigid" switch if the display contained a group of dots that was moved together rigidly, and to press the "non-rigid" switch otherwise. A group of dots was defined as moving together rigidly if "at least four dots maintain constant distances from each other regardless of how the entire group moves."

Results

A d' was computed for each subject and stimulus condition (Table 3). Of 80 d' s, 48 were significantly different from zero ($p < .05$). For zero noise points, 15 (of 16) d' s were significantly different from zero. For four

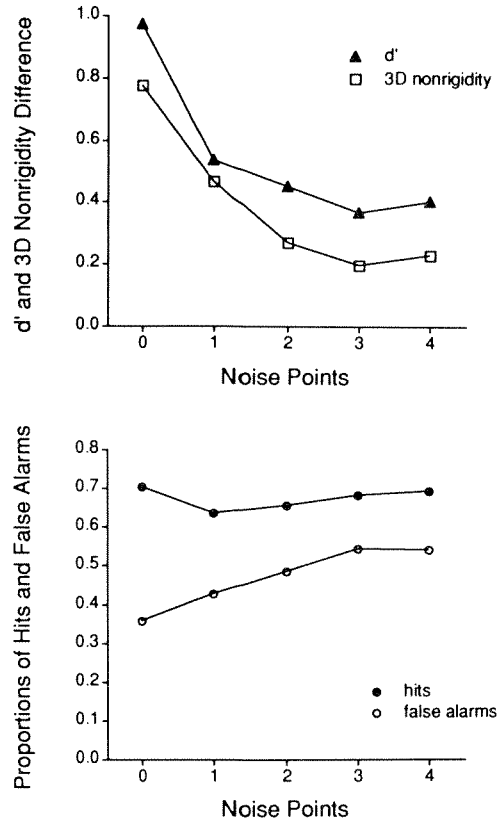


Figure 2. d' , difference in 3-D nonrigidity (noise nonrigidity - signal nonrigidity), proportion of hits, and proportion of false alarms as functions of the number of noise points in the signal displays (Experiment 3).

noise points, 7 (of 16) d' s were significantly different from zero.

The independent variables in the ANOVA were number of noise points and number of views. There were two significant effects. The main effect of number of noise points [$F(4,12) = 26.79, p < .01, \omega^2 = 0.34$] showed a decrease in d' with more noise points. The mean d' values for 0, 1, 2, 3, and 4 noise points were 0.97, 0.54, 0.45, 0.37, and 0.40, respectively. Post hoc comparisons showed only the differences between zero noise points and nonzero noise point conditions to be significant. The main effect of number of views [$F(3,9) = 10.43, p < .01, \omega^2 = 0.21$] showed an increase in d' with greater numbers of views. The mean d' s for 2, 3, 4, and 12 views were 0.34, 0.52, 0.49, and 0.84, respectively. Post hoc comparisons showed only the differences between 12 views and smaller numbers of views to be significant.

In the previous experiments, we examined the relationship between accuracy of discrimination and a measure of 3-D nonrigidity for the noise trials. For those experiments, the 3-D nonrigidity for the signal trials was always zero. In Experiment 3, 3-D nonrigidity increased for the signal trials as additional noise points were added.

Table 3
 d' Scores in Experiment 3

Subject	Number of Noise Points				
	0	1	2	3	4
Two Views					
F.	0.740*	0.420	0.170	0.125	0.645*
M.	0.300	0.545*	0.000	-0.135	0.000
G.	0.695*	0.320	0.895*	0.000	0.105
L.	0.740*	0.380	0.555*	-0.045	0.305
Three Views					
F.	1.200*	0.815*	0.725*	0.160	0.320
M.	0.630*	0.505*	0.245	0.175	0.090
G.	0.550*	0.445	0.730*	0.490*	0.305
L.	1.045*	0.515*	0.310	0.385	0.715*
Four Views					
F.	0.950*	0.595*	0.465*	0.415	0.375
M.	1.040*	0.505*	0.260	0.565*	0.530*
G.	0.505*	0.375	-0.205	0.650*	0.510*
L.	0.940*	0.595*	0.275	0.180	0.220
Twelve Views					
F.	1.560*	0.850*	0.695*	0.945*	0.815*
M.	2.005*	0.865*	0.870*	0.480*	0.660*
G.	1.345*	0.465*	0.685*	0.660*	0.250
L.	1.330*	0.375	0.550*	0.800*	0.555*

* $p < .05$.

It is likely that discriminability in this experiment was based on a relationship between 3-D nonrigidity in the signal trials and 3-D nonrigidity in the noise trials. We examined two obvious relationships: the ratio of the nonrigidity measure (signal trials/noise trials) and the difference in the measure (noise trials - signal trials). The correlations with d' , across the 20 combinations of views and noise points, were $-.65$ for the ratio measure and $.87$ for the difference measure. We therefore present the difference measure in Figures 2 and 3. Figure 2 shows the effects of number of noise points on d' and on the difference between noise and signal trials in 3-D nonrigidity. The hit rate and false-alarm rate are also shown. Figure 3 presents these effects as the number of views increases from 2 to 12. These results suggest that the difference in nonrigidity, or some related quantity, accounts both for the effects of points and for the effects of views. These effects are due primarily to changes in the false-alarm rate.

GENERAL DISCUSSION

On the basis of the rigidity constraint alone, human observers can discriminate rigid motion from nonrigid motion at the minimum level of points and views at which

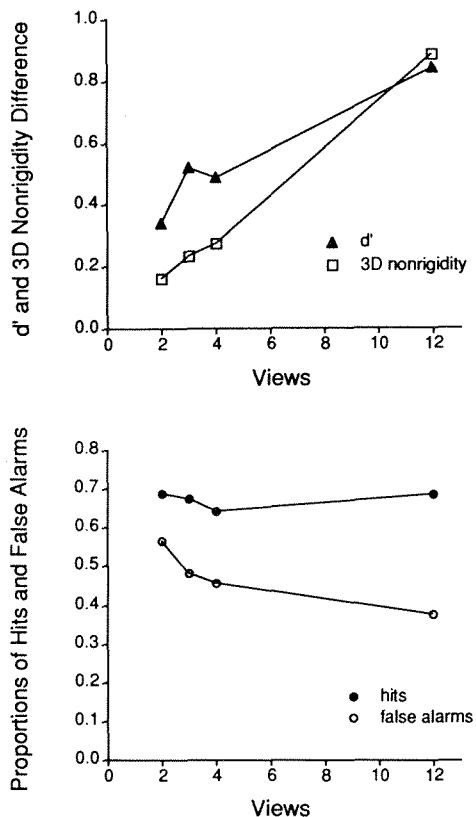


Figure 3. d' , difference in 3-D nonrigidity (noise nonrigidity - signal nonrigidity), proportion of hits, and proportion of false alarms as functions of the number of views (Experiment 3).

such discrimination is theoretically possible: two views of four points. For discriminations between displays in which all points were either moving rigidly or rotating about separate axes, accuracy depended on the deviation of the nonrigid displays from rigid motion. Our measure of this deviation, the mean across pairs of points of the variance in the interpoint distance over views, was related to the discriminability of rigid from nonrigid displays. This measure is based on the 3-D structure used to generate the displays. The usefulness of this measure is especially interesting in the case of the two-view displays, because the same two-view displays can be generated from an infinite number of rigid 3-D structures (Bennett, Hoffman, Nicola, & Prakash, 1989).

Increasing the number of points in a rigidly moving group does not lead to a clear increase in accuracy, although there was a nonsignificant increase from four to more than four points. It is certainly possible that an effect of points would be found for larger numbers of points—numbers sufficient to give the configuration a clear shape. Increasing the number of views did increase accuracy of discrimination, but this can be attributed to the increase in nonrigidity of the nonrigid displays. With points rotating about separate axes, the variance of the distances between pairs of points increases with number of views. Our measure of 3-D nonrigidity, based on these variances, correlated $.985$ with d' across the five levels of views.

Although human subjects can discriminate rigid from nonrigid structures at the minimum level of points and views at which this discrimination is theoretically possible, accuracy drops sharply when even one point that is not part of the rigid structure is added to a rigid display. It appears that human observers are not proficient at analyses that require testing subgroups of points to determine whether one subgroup is present that is moving rigidly. (With five points there would be five such subgroups to test. This may not seem to be much of a processing load from a computational viewpoint, but five subgroups involving six distances each in one display may be difficult for human subjects to process.) These results may appear to be in conflict with Ullman's (1979) well-known demonstration that two concentric cylinders differing in diameter are easily segregated by the human visual system. Ullman's stimuli, however, are not directly comparable to the present stimuli. Ullman used a large number of points and views, not the minimal numbers used in the present research. Perhaps more importantly, the motion in the demonstration was rotation about a fixed axis at a constant angular velocity. Bennett and Hoffman (1985) have shown that a fixed-axis constraint is sufficient mathematically for recovering 3-D structure from four orthographic views of two points or three orthographic views of four points; a rigidity constraint is not necessary. Demonstrations by Braunstein (1983) and Ramachandran, Cobb, and Rogers-Ramachandran (1988) also indicate that the perceptual segmentation of two rotating cylinders may not be based entirely on the use of a rigidity constraint.

The sharp drop in accuracy in detecting the presence of a rigid structure when noise points were added to the structure is consistent with Lappin et al.'s (1980) results with larger numbers of dots. In that study, accuracy in determining which of two displays had more coherent motion was highest when one of the displays was completely rigid, but dropped sharply when both displays contained nonrigid motion. If the subjects in the present experiments were primarily engaged in detecting nonrigid motion, rather than detecting rigid groups of points, it is not surprising that accuracy should have dropped sharply when both the signal trials and noise trials included nonrigid motion.

Discrimination between rigid and nonrigid structures, at least on the basis of small numbers of points and views, does not appear to be an easy task for human subjects. Subjective reports indicate that this task requires careful attention. It is possible that the task could be performed with less effort if the nonrigid motions differed even more from the rigid motions. In our displays, the same center of rotation was used for all points, whether or not they were part of a rigid structure. Generically, feature points that are moving independently would probably not have the same center of rotation. This probably made discriminations especially difficult in the present study, but it was necessary, to prevent a consistent relationship between nonrigidity in the 2-D projection and nonrigidity in 3-D.

In presenting a signal-detection analysis of the present experiments, we chose to define displays containing groups of at least four points moving together rigidly as signal displays, and displays lacking such rigid groups as noise displays. Our results suggest that the opposite interpretation may be worth considering. Discrimination of rigid motion from nonrigid motion may be conceived of as detecting deviations from constant interpoint distances in 3-D—that is, as detecting nonrigidity. Thus, in Experiments 1 and 2, the rigid displays might have been defined as the “noise displays” and the nonrigid displays as the “signal-plus-noise displays.” Increasing the 3-D nonrigidity of the nonrigid displays by increasing the number of views in Experiment 2 could then be described as increasing the signal strength, with the expected result of increasing d' . In Experiment 3, subjects may have been discriminating between levels of nonrigidity (i.e., between two levels of signal) rather than detecting rigid groups. Introspective reports suggest that subjects were both looking for rigid groups and looking for deviations from rigidity. The relationship between signal detection concepts and the discrimination of rigid from nonrigid motion would be worth exploring further with additional experimental manipulations.

In conclusion, these experiments reveal that human subjects are surprisingly good at some aspects of analyzing 3-D structures and surprisingly poor at others. Human subjects can discriminate rigid from nonrigid motion at exactly the minimum levels of points and views specified by theoretical analyses, suggesting that such analyses may be of relevance to the study of human vision. But when

the task is changed to determining whether a rigid structure is present in noise, performance falls off sharply with even one noise point. We need to look further into the issue of whether a rigidity constraint is useful in perceptual grouping, or whether other constraints must determine grouping before a rigidity constraint can be applied.

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NOTES

1. Points move rigidly if all of their 3-D interpoint distances remain constant over time.

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