Runs in a Ring
Author(s): D. E. Barton and F. N. David
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For fixed r the vectors $\mathbf{x}$ and $\mathbf{y}$ in the sample space are at a fixed angle θ to each other and otherwise randomly orientated in the space $\Sigma x_i = 0$, whatever the value of ρ. Consider $E\{\text{sgn}(x_1 - x_2)\,\text{sgn}(y_1 - y_2) \mid r\}$. $\text{sgn}(x_1 - x_2)\,\text{sgn}(y_1 - y_2)$ is $+1$ if $\mathbf{x}$, $\mathbf{y}$ are both on the same side of the hyperplane $x_1 - x_2 = 0$, and -1 in the opposite case. For any fixed orientation of the two-dimensional plane spanned by $(\mathbf{x}, \mathbf{y})$ the hyperplane $x_1 - x_2 = 0$ cuts it in a line, and the probability that this line lies between $\mathbf{x}$ and $\mathbf{y}$ is $\theta/\pi = \cos^{-1} r/\pi = p$, say. This is therefore the probability, for fixed r, that $\mathbf{x}$ and $\mathbf{y}$ are on opposite sides of $x_1 - x_2 = 0$ for random orientations of $(\mathbf{x}, \mathbf{y})$. Hence

$$E\{\text{sgn}(x_1 - x_2)\,\text{sgn}(y_1 - y_2) \mid r\} = -p + (1-p) = 1 - 2p$$
$$= 1 - (2/\pi)\cos^{-1} r = (2/\pi)\sin^{-1} r$$

and the result (1) follows.

3. An immediate consequence is that for any subsample $n' < n$ from $(x_1, y_1) \ldots (x_n, y_n)$ the rank correlation coefficient t' (the sample value of τ) is such that

$$E(t' \mid r) = (2/\pi)\sin^{-1} r. \tag{5}$$

If we let n tend to infinity we find the familiar result

$$E(t' \mid \rho) = (2/\pi)\sin^{-1} \rho. \tag{6}$$

It also follows, on taking $n' = n$, that the regression of t on r is given by

$$E(t \mid r) = (2/\pi)\sin^{-1} r \tag{7}$$

whatever the value of ρ.

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Runs in a ring

BY D. E. BARTON AND F. N. DAVID

University College London

Some time ago when we were working on the distribution and properties of runs of multiple colours in a line we worked out the corresponding theory for runs in a ring, but were discouraged from publication by the criticisms that essentially no new mathematical points were involved and that there was no obvious statistical application. We do not agree with these criticisms, and the recent paper by Dawson & Good (1958) indicates that there is some interest in the runs in a ring problem. Accordingly, we give here a summary of results and some tables.

r identical beads of k colours are supposed in a random order. The problem of the ring has two facets. First, the ring may be supposed to have been built up by sampling randomly from a finite population of beads, the beads being strung on a thread when selected and the ends of the thread tied after r beads. Secondly, a handful of beads $r_1, r_2, \ldots, r_k$ $(\Sigma r_i = r)$ can be imagined as placed in a circle in which case symmetries will need to be allowed for.

The first problem where the ring is the line bent to a circle can be solved either from first principles—Whitworth (1886) gives a method—or by adopting the method for the line. If there are T runs in the line, i.e. $T-1$ alternations of colour, there will be $T-1$ runs in the ring if the same colour is at the beginning and the end of the line, and T otherwise. The appropriate multinomial term and the number of permutations are given for $r = 2, 3, \ldots, 12$ and two, three and four colours in Table 1. If $S = r - T$ the mth factorial moment of S is just $r/(r-m)$ times the mth factorial moment of the same statistic calculated for the line (Barton & David (1957)). The same limits, the normal and Poisson, hold under the same conditions, and the positive binomial with the correct first two moments is again a suitable approximating function. Given that the beads are not random in the ring, but that the probabilities are those of a simple Markoff chain, the distribution of T under this hypothesis may be derived on precisely the same lines set out by David (1947) for the two-colour-line case.

Table 1*a*. (*Two colours.*) *Ring permutations in repeated sampling*

(Probabilities are obtained by dividing the number of permutations by the appropriate multinomial term.)

Multinomial term	r	Partition	Runs					
			2	4	6	8	10	12
2	2	(1^2)	2					
3	3	(21)	3					
4	4	(31)	4					
6	4	(2^2)	4	2				
5	5	(41)	5					
10	5	(32)	5	5				
6	6	(51)	6					
15	6	(42)	6	9				
20	6	(3^2)	6	12	2			
7	7	(61)	7					
21	7	(52)	7	14				
35	7	(43)	7	21	7			
8	8	(71)	8					
28	8	(62)	8	20				
56	8	(53)	8	32	16			
70	8	(4^2)	8	36	24	2		
9	9	(81)	9					
36	9	(72)	9	27				
84	9	(63)	9	45	30			
126	9	(54)	9	54	54	9		
10	10	(91)	10					
45	10	(82)	10	35				
120	10	(73)	10	60	50			
210	10	(64)	10	75	100	25		
252	10	(5^2)	10	80	120	40	2	
11	11	(10, 1)	11					
55	11	(92)	11	44				
165	11	(83)	11	77	77			
330	11	(74)	11	99	165	55		
462	11	(65)	11	110	220	110	11	
12	12	(11, 1)	12					
66	12	(10, 2)	12	54				
220	12	(93)	12	96	112			
495	12	(84)	12	126	252	105		
792	12	(75)	12	144	360	240	36	
924	12	(6^2)	12	150	400	300	60	2

Table 1*b*. (*Three colours.*) *Ring permutations in repeated sampling*

(Probabilities are obtained by dividing the number of permutations by the appropriate multinomial term.)

Multinomial term	r	Partition	Runs 3	4	5	6	7	8	9	10	11	12
6	3	(1^3)	6									
12	4	(21^2)	8	4								
20	5	(31^2)	10	10								
30	5	$(2^2\,1)$	10	10	10							
30	6	(41^2)	12	18								
60	6	(321)	12	18	24	6						
90	6	(2^3)	12	18	36	24						
42	7	(51^2)	14	28								
105	7	(421)	14	28	42	21						
140	7	$(3^2\,1)$	14	28	56	28	14					
210	7	(32^2)	14	28	70	70	28					
56	8	(61^2)	16	40								
168	8	(521)	16	40	64	48						
280	8	(431)	16	40	96	72	48	8				
420	8	(42^2)	16	40	112	144	96	12				
560	8	$(3^2\,2)$	16	40	128	176	144	56				
72	9	(71^2)	18	54								
252	9	(621)	18	54	90	90						
504	9	(531)	18	54	144	144	108	36				
756	9	(52^2)	18	54	162	252	216	54				
630	9	$(4^2\,1)$	18	54	162	162	162	54	18			
1,260	9	(432)	18	54	198	333	378	225	54			
1,680	9	(3^3)	18	54	216	396	486	378	132			
90	10	(81^2)	20	70								
360	10	(721)	20	70	120	150						
840	10	(631)	20	70	200	250	200	100				
1,260	10	(62^2)	20	70	220	400	400	150				
1,260	10	(541)	20	70	240	300	360	180	80	10		
2,520	10	(532)	20	70	280	550	760	580	240	20		
4,200	10	(43^2)	20	70	320	700	1,100	1,130	680	180		
3,150	10	$(4^2\,2)$	20	70	300	600	900	780	380	100		
110	11	(91^2)	22	88								
495	11	(821)	22	88	154	231						
1,320	11	(731)	22	88	264	396	330	220				
1,980	11	(72^2)	22	88	286	594	660	330				
2,310	11	(641)	22	88	330	495	660	440	220	55		
4,620	11	(632)	22	88	374	836	1,320	1,210	660	110		
2,772	11	$(5^2\,1)$	22	88	352	528	792	528	352	88	22	
6,930	11	(542)	22	88	418	957	1,716	1,848	1,276	517	88	
9,240	11	(53^2)	22	88	440	1,100	2,046	2,508	2,024	880	132	
11,550	11	$(4^2\,3)$	22	88	462	1,188	2,310	3,058	2,662	1,408	352	
132	12	$(10, 1^2)$	24	108								
660	12	(921)	24	108	192	336						
1,980	12	(831)	24	108	336	588	504	420				
2,970	12	(82^2)	24	108	360	840	1,008	630				
3,960	12	(741)	24	108	432	756	1,080	900	480	180		
7,920	12	(732)	24	108	480	1,200	2,088	2,220	1,440	360		
5,544	12	(651)	24	108	480	840	1,440	1,200	960	360	120	12
13,860	12	(642)	24	108	552	1,416	2,880	3,630	3,120	1,620	480	30
18,480	12	(63^2)	24	108	576	1,608	3,384	4,740	4,640	2,640	720	40
16,632	12	$(5^2\,2)$	24	108	576	1,488	3,168	4,176	3,840	2,304	792	156
27,720	12	(543)	24	108	624	1,812	4,104	6,396	7,008	5,076	2,160	408
34,650	12	(4^3)	24	108	648	1,944	4,536	7,506	8,712	6,912	3,456	804

Table 1*c*. (*Four colours.*) *Ring permutations in repeated sampling*

(Probabilities are obtained by dividing the number of permutations by the appropriate multinomial term.)

Multinomial term	r	Partition	Runs 4	5	6	7	8	9	10	11	12
24	4	(1^4)	24								
60	5	(21^3)	30	30							
120	6	(31^3)	36	72	12						
180	6	$(2^2\,1^2)$	36	72	72						
210	7	(41^3)	42	126	42						
420	7	(321^2)	42	126	182	70					
630	7	$(2^3\,1)$	42	126	252	210					
336	8	(51^3)	48	192	96						
840	8	(421^2)	48	192	336	240	24				
1,120	8	$(3^2\,1^2)$	48	192	416	320	144				
1,680	8	$(32^2\,1)$	48	192	496	640	304				
2,520	8	(2^4)	48	192	576	960	744				
504	9	(61^3)	54	270	180						
1,512	9	(521^2)	54	270	540	540	108				
2,520	9	(431^2)	54	270	720	810	540	126			
3,780	9	$(42^2\,1)$	54	270	810	1,350	1,080	216			
5,040	9	$(3^2\,21)$	54	270	900	1,620	1,530	666			
7,560	9	(32^3)	54	270	990	2,160	2,700	1,386			
720	10	(71^3)	60	360	300						
2,520	10	(621^2)	60	360	800	1,000	300				
5,040	10	(531^2)	60	360	1,100	1,600	1,320	560	40		
7,560	10	$(52^2\,1)$	60	360	1,200	2,400	2,520	960	60		
6,300	10	$(4^2\,1^2)$	60	360	1,200	1,800	1,800	840	240		
12,600	10	(4321)	60	360	1,400	3,100	4,050	2,840	790		
18,900	10	(42^3)	60	360	1,500	3,900	6,300	5,340	1,440		
16,800	10	$(3^3\,1)$	60	360	1,500	3,600	5,100	4,440	1,740		
25,200	10	$(3^2\,2^2)$	60	360	1,600	4,400	7,700	7,640	3,440		
990	11	(81^3)	66	462	462						
3,960	11	(721^2)	66	462	1,122	1,650	660				
9,240	11	(631^2)	66	462	1,562	2,750	2,640	1,540	220		
13,860	11	$(62^2\,1)$	66	462	1,672	3,850	4,840	2,640	330		
13,860	11	(541^2)	66	462	1,782	3,300	4,092	2,772	1,188	198	
27,720	11	(5321)	66	462	2,002	5,170	8,272	7,612	3,652	484	
41,580	11	(52^3)	66	462	2,112	6,270	12,012	13,332	6,534	792	
34,650	11	$(4^2\,21)$	66	462	2,112	5,610	9,570	9,834	5,478	1,518	
46,200	11	$(43^2\,1)$	66	462	2,222	6,380	11,550	13,574	9,218	2,728	
69,300	11	(432^2)	66	462	2,332	7,480	16,060	22,604	16,258	5,038	
92,400	11	$(3^3\,2)$	66	462	2,442	8,250	18,810	27,654	24,618	10,098	
1,320	12	(91^3)	72	576	672						
5,940	12	(821^2)	72	576	1,512	2,520	1,260				
15,840	12	(731^2)	72	576	2,112	4,320	4,680	3,360	720		
23,760	12	$(72^2\,1)$	72	576	2,232	5,760	8,280	5,760	1,080		
27,720	12	(641^2)	72	576	2,472	5,400	7,740	6,720	3,600	1,080	60
55,440	12	(6321)	72	576	2,712	7,920	14,640	16,320	10,440	2,640	120
83,160	12	(62^3)	72	576	2,832	9,360	20,340	27,120	18,360	4,320	180
33,264	12	$(5^2\,1^2)$	72	576	2,592	5,760	8,928	8,064	5,184	1,728	360
83,160	12	(5421)	72	576	2,952	9,000	18,324	23,616	18,576	8,424	1,620
110,880	12	$(53^2\,1)$	72	576	3,072	10,080	21,624	30,816	28,080	14,016	2,544
166,320	12	(532^2)	72	576	3,192	11,520	28,584	46,656	46,584	24,768	4,368
138,600	12	$(4^2\,31)$	72	576	3,192	10,800	24,120	36,768	36,144	21,168	5,760
207,900	12	$(4^2\,2^2)$	72	576	3,312	12,240	31,500	54,288	59,184	36,288	10,440
277,200	12	$(43^2\,2)$	72	576	3,432	13,320	36,060	66,528	80,784	58,008	18,420
369,600	12	(3^4)	72	576	3,552	14,400	41,040	80,448	107,424	88,128	33,960

The second problem discussed partially by Jablonski (1892) is probably the one which Whitworth really had in mind. Jablonski imagined r_i $(i = 1, 2, \ldots, k)$ beads of k different colours set down in a ring. He enumerated the total number of *different* arrangements which could be made of these beads allowing for rotations and symmetries but not for turning the ring over. Except where there are common factors among $\{r_i\}$ the total number of arrangements will be

$$(r-1)! \Big/ \prod_{i=1}^{k} r_i!.$$

Further, the distribution of the number of runs will be the same as the distribution of the number of runs in the linear problem with the common factor r cancelled out. We show here that Jablonski's method for the enumeration of different arrangements in the ring may be shortened and extended to give the distribution of runs also. For clarity we will refer to these runs as Jablonski runs.

Consider any given arrangement in the ring which we shall call a ring permutation, A. There will be r linear arrangements of A say $A_1, A_2, \ldots, A_r$, which we may obtain by cutting the ring at the r possible points. The set of these, which we will call $S(A)$ consists of the r cyclic permutations of any one of them and we will let $S(A)$ contain just m_i different linear permutations. If d_i is one of the divisors of the highest common factor h, say, of $r_1, r_2, \ldots, r_k$, then $m_i = r/d_i$. Suppose there are n divisors and let

$$d_1 = 1, \ldots, d_n = h.$$

Further, let A_1 be that member of $S(A)$ consisting of the juxtaposition of d_i similar arrays of m_i beads with

$$r_1/d_i, r_2/d_i, \ldots, r_k/d_i$$

of the respective colours. If $Q_{d_i}.\, d_i/r$ is the number of ring permutations which give linear arrays with just m_i different line permutations, then

$$\sum_{d_i|d|h} Q_d = \frac{(r/d_i)!}{(r_1/d_i)!\ldots(r_k/d_i)!} = T_{d_i}(\text{say}) \quad (i = 1, 2, \ldots, n)$$

These n linear equations for the $\{Q_d\}$ have a unique solution so that the total number of Jablonski arrangements is

$$J = \frac{1}{r} \sum_{d|h} d.Q_d.$$

Let $\phi(d)$ denote Euler's ϕ-function, i.e. if d is a positive integer, $\phi(d)$ will denote the number of positive integers not exceeding d which are relatively prime to d. We have that

$$\sum_{j=1}^{n} \phi(d_j)\, T_{d_j} = \sum_{d\,|h} \sum_{d_j|d|h} \phi(d_j)\, Q_d = \sum_{d|h} Q_d \sum_{d_j|d} \phi(d_j) = \sum_{d|h} dQ_d = rJ.$$

Now let $Q_d(t)\, d/r$ be the number of those ring permutations with linear arrangements containing just r/d line permutations which have t ring runs; let A' be such a ring permutation and A_1' a corresponding line permutation. A_1' consists of d similar line arrays each of which is an unrolling of a ring permutation of r/d beads of colour composition $(r_1/d, \ldots, r_k/d)$ and this ring permutation has t/d ring runs. If $p_{d_i}(t)$ is the proportion of permutations of elements $(r_1/d_i, \ldots, r_k/d_i)$ with r/d_i ring runs where repetitions are allowed, then

$$\sum_{d_i|d|h} Q_d(t) = T_{d_i} p_{d_i}(t)$$

and

$$J(t) = \sum_{d|h} Q_d(t) = \frac{1}{r} \sum_{d|h} \phi(d)\, T_d p_d(t)$$

is the number of ring runs with repeated ring permutations not allowed.

A worked example will illustrate the method given in the previous section. Suppose $r = 12$ and we have three colours $(6, 4, 2)$. It is seen that $h = 2$ so that

$$Q_1 + Q_2 = 13{,}860, \quad Q_2 = 60$$

and

$$J = 1150 + 10 = 1160.$$

Or alternatively since

$$\phi(1) = 1 = \phi(2)$$

we have

$$J = \tfrac{1}{12}(13860 + 60) = 1160.$$

Further, $T_1 p_1(t)/r$ takes the values 2, 9, 46, 118, 240, $302\frac{1}{2}$, 260, 135, 40, $2\frac{1}{2}$ as t runs through the integers 3, 4, 5, ..., 12, whilst $T_2 p_2(t)/r$ takes the values 1, $1\frac{1}{2}$, 2, $\frac{1}{2}$ at the values 6, 8, 10 and 12 and is zero elsewhere. Thus $J(t)$ takes the values 2, 9, 46, 119, 240, 304, 260, 137, 40, 3 with a total of 1160 ($= J$) as expected. Values of $J(t)$ are given in Table 2 for those partitions of $r \leqslant 12$, where the highest common factor of the parts is 2 or more than 2. Table 3 gives the values of Euler's ϕ-function necessary for the enumeration of ring-runs up to and including $r = 12$.

We may, if we choose, regard the different ring permutations as forming a fundamental probability set. The jth moment of the distribution of runs is given by

$$\mu'_j = \sum_t J(t)\, t^j / J.$$

Table 2*a*. *Jablonski runs.* (*Two-colours.*)

(Only those distributions are given which differ from those of Table 1 divided by r.)

Total	r	Partition	Runs 2	4	6	8	10	12
2	4	(2^2)	1	1				
3	6	(42)	1	2				
4	6	(3^2)	1	2	1			
4	8	(62)	1	3				
10	8	(4^2)	1	5	3	1		
10	9	(63)	1	5	4			
5	10	(82)	1	4				
22	10	(64)	1	8	10	3		
26	10	(5^2)	1	8	12	4	1	
6	12	(10, 2)	1	5				
19	12	(93)	1	8	10			
43	12	(84)	1	11	21	10		
80	12	(6^2)	1	13	34	26	5	1

Table 2*b*. *Jablonski runs.* (*Three-colours.*)

Total	r	Partition	Runs 3	4	5	6	7	8	9	10	11	12
16	6	(2^3)	2	3	6	5						
54	8	(42^2)	2	5	14	19	12	2				
188	9	(3^3)	2	6	24	44	54	42	16			
128	10	(62^2)	2	7	22	41	40	16				
318	10	$(4^2, 2)$	2	7	30	61	90	79	38	11		
250	12	(82^2)	2	9	30	71	84	54				
1160	12	(642)	2	9	46	119	240	304	260	137	40	3
1542	12	(63^2)	2	9	48	134	282	395	388	220	60	4
2896	12	(4^3)	2	9	54	163	378	627	726	579	288	70

Table 2c. *Jablonski runs.* (*Four-colours.*)

Total	r	Partition	Runs								
			4	5	6	7	8	9	10	11	12
318	8	(2^4)	6	24	72	120	96				
1,896	10	(42^3)	6	36	150	390	633	534	147		
6,940	12	(62^3)	6	48	236	780	1,698	2,260	1,536	360	16
17,340	12	$(4^2 2^2)$	6	48	276	1,020	2,628	4,524	4,938	3,024	876
30,804	12	(3^4)	6	48	296	1,200	3,420	6,704	8,952	7,344	2,834

Table 3. *Values of Euler's ϕ-function for enumerating permutations for values of* $r \leqslant 12$

d	1	2	3	4	5	6
$\phi(d)$	1	1	2	2	4	2

Simple formulae do not flow from this expression but we note that if M_1 is the mean number of runs when repetitions are allowed then

$$\mu_1' = M_1 + \frac{M_1}{rJ} \sum_{d|h} \binom{d-1}{r-d} \phi(d)\, T_d.$$

It is seen that

$$\mu_1' \geqslant M_1$$

with equality if, and only if, the highest common factor of $(r_1, \ldots, r_k)$ is unity.

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Some applications of Meijer-G functions to distribution problems in statistics

BY D. G. KABE

Karnatak University, Dharwar (India)

1. Although the Fourier transform is recognized to be a powerful tool in statistical distribution theory, the Mellin transform seems to have been neglected. Epstein (1948) has used Mellin transforms to derive certain univariate distribution functions and Nair (1939) has indicated their applications to some multivariate problems. The Mellin transform of the frequency function $f(x)$, $(0 < x < \infty)$, of a random variable X, is defined to be

$$g(s) = \int_0^\infty x^{s-1} f(x)\, dx, \tag{1}$$

the inverse transform being

$$f(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} x^{-s} g(s)\, ds. \tag{2}$$